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The supersymmetric sigma model, topological quantum mechanics and knot invariants

J.M. GUILARTE (*)

Departemento de Fisica,
Facultad de Ciencias,
Salamanca 37.008, Spain

Abstract. *A field theoretical discussion of the complex constructed by Floer in order to prove the Arnold conjecture concerning a Morse theory for the fixed point set of diffeomorphisms of a symplectic manifold is offered. The approach is modelled on the supersymmetric non-linear sigma model in* $(1 + 1)$ *-space-time dimensions. The associated topological quantum mechanics are proved to be related with the Lefschetz formula of fixed points of holomorphic maps on complex manifolds. Expectation values of vertex operators provide a quantum mechanical version of the Witten-Jones invariants of knot theory. Stochastic quantization of the topological quantum mechanics is performed in the setting of the supersymmetric sigma model. Expectation values of significant operators in topological quantum mechanics are, non-trivially, obtained as the average, large stochastic time behaviour, of similar observables in the supersymmetric sigma model. Applications of the supersymmetric sigma model to the physics of liquid crystal materials are suggested. A topological theory of spin for relativistic particles moving in curved space-time arises from topological quantum mechanics.*

1. INTRODUCTION AND CONCLUSIONS

In the past few years we have witnessed spectacular developments in topology which can be approached in the framework of quantum field theory [1]. The field theoretical viewpoint being non-rigorous, in contrast to the original mathematical work on geometry

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of low dimensional manifolds [2], strong reasons should be given to extend the set of models where Witten ideas are to be applied. These are twofold: on the mathematical side, the scope of the intersection between field theory and mathematics is broadening to become a whole area including quantum physics and topology as new features. From the physical side, the models entering the game, gauge theories and sigma models, are so important that the unveiling of any hidden subtle structure must be welcomed. A general picture of the modles in QFT with topological significance, as described in the last paper of Ref. [1] is summarized in Fig. 1 with special attention to the dual rôle of the Hamiltonian and Lagrangian formalisms. In this paper we will focus in models in spatial-dimension $d = 1$ and $d = 0$ in the Hamiltonian formalism, placing special emphasis on developing the link between them provided by the stochastic quantization of the $d = 0$-dimensional model.

1. We shall first describe Floer's proof of the Arnold conjecture in the setting of a (non-relativistic) supersymmetric extension of the non-linear sigma model in $(1 + 1)$space-time dimensions. Floer proved the Arnold conjecture [4] concerning a Morse Theory for fixed points of diffeomorphisms of a symplectic manifold by defining the homology of the boundary operator obtained from the holomorphic curves (instantons) bounded by intersecting Lagrangian submanifolds. This is close in spirit to the Smale version [5] of Morse Theory inspired in classical mechanics. Our presentation will be along the lines put forward by Witten in the first, seminal, paper of Reference [1], essentially the semi-classical approximation to quantum mechanics, closely following a parallel development of the topological implications of Supersymmetric Yang-Mills Theory [6]. Floer's Theory is a Morse Theory for the space of maps from the unit interval to a symplectic manifold M with endpoints in Lagrangian submanifolds. The Morse functional is the area functional subtended by a path between intersecting Lagrangian submanifolds.

2. A very intriguing property of the Floer Hamiltonian is the existence of an obvious zero energy eigenfunctional: the exponential of the area functional [7]. It is however non-normalizable but if one computes the norm by the steepest descent method one finds that the origin of the divergences is the Dirac sea tied to all the candidates, before tunnelling, to become ground states. For this reason we shall call it the global ground state, even though the name can be misleading owing to its physical meaning: the ground states should be normalizable! In any case the (complex) norm carries topological information.

By analytic continuation to the imaginay axis of the deformation parameter the norm of the ground state can be seen as the partition function for a model in quantum mechanics for which the action is the area functional. Because the corresponding Hamiltonian is zero there are no dynamics and we are dealing with topological quantum mechanics. In fact, the partition function must be an invariant of the target manifold. We will compute this in the case where M is a Kähler manifold to meet the Ledfschetz fixed

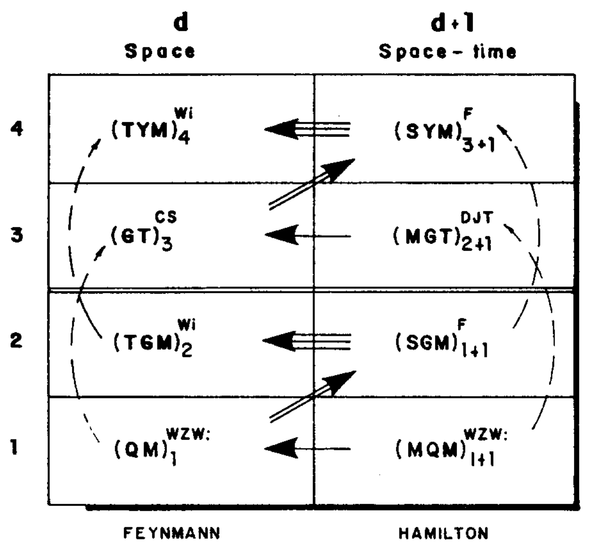

Fig. 1. In this Figure we show a Table summarizing the models in QFT we are going to deal with throughout the paper. Essentially they are sigma models in one-and two-space or space-time dimensions. Distinguishing a time-coordinate amounts to choosing the Hamiltonian, as opposed to the Lagrangian or Feymann, quantization procedure. We will focus on the Supersymmetric Sigma Model in $(1+1)$-dimensions, $(S\sigma M)_{1+1}^F$, and on the Wess-Zumino-Witten Quantum Mechanics in 1-dimension; $(QM)_1^{WZW}$ is the infinite mass limit of massive Quantum Mechanics, $(MQM)_{0+1}^{WZW}$, single arrow, where the interaction term in the Lagrangian is the Wess-Zumino term. It is possible to consider a covariant stochastic time to end in the Topological Sigma Model of Witten, $(T\sigma M)_2^{W_i}$, triple arrow. The choice of a particular quantization procedure for each model assumed in the Table is the most natural one. There is a parallel situation for gauge theories in three-and four-space and space-time dimensions as meant by the broken arrows and the dictionary:

$(MGT)_{2+1}^{DJT}$: **Massive Gauge Theory.**
$(GT)_2^{CS}$: **Chern-Simmons Gauge Theory.**
$(SYM)_{3+1}^F$: **Supersymmetric Yang-Mills Theory.**
$(TYM)_4^{W_i}$: **Topological Yang-Mills Theory.**

The purpose of this paper is to explain the mathematical and physical implications of the quantization of the sigma models in the Table and the links between them.

point formula for the Lefschetz number of the Dolbeault complex. To obtain the stationary phase approximation of the partition function we will need to fix some «gauge» freedom, the corresponding Fadeev-Popov ghosts supplying a fermionic sector in the model. It happens that our model is the limit of the sigma model in supersymmetric Quantum Mechanics appearing in most recent conventional proofs of the index theorem

[8], where the non-zero eigenvalues of the Hamiltonian decouples by going to infinity.

3. One of the main themes of this paper is to establish a link between topological quantum mechanics and the supersymmetric sigma model in $(1+1)$-dimensions: the stochastic quantization of a system for which the actions is the area functional is performed by means of the second system taking as the Langevin equation the gradient flow equation [9]. This is a fairly general fact working also in supersymmetric Quantum Mechanics and Yang-Mills Theory whose topological associated models are zero-dimensional «potential» theory and three-dimensional Chern-Simons Theory, in Lagrangian formalism, respectively. Stochastic quantization provides a bridge between topological modles in d-and $(d+1)$-dimensions and sheds light on the problem of understanding the different rôle of the area functional, or similar Morse functionals, in $(0+1)$-and $(1+1)$-dimensions. In contrast with theories where the ground state is unique, the stochastic quantization of topological quantum mechanics has special characteristcs owing to the lack of ergodicity. We shall show that the large time behaviour of the tunneling amplitudes between local ground states in the supersymmetric sigma model gives the factors of the expansion of the partition function of topological quantum mechanics around the critical points of the action functional. The obvious topological relationship between Floer homology and special critical points arises as the large-time behaviour of the tunnel effect.

It is also interesting to observe that stochastic quantization can be interpreted as the $BRST$ gauge fixing of a purely topological action. This is the deep reason for the essentially topological content of some supersymmetric models in Quantum Field Theory.

4. Blip-vertex operators, of the kind introduced in soliton physics [10], are very important in topological quantum mechanics. They «pierce» the unit interval of euclidean time creating n new intervals and in, this sense, TQM is a topological model defined on a space of n points. Its expectation values are invariants for braids, maps from [0, 1] to the set of n unordered points in C which are equal at 0 and 1. Because a knot is the closure of a link they also provide knot invariants. In fact, this contact with the Chern-Simons theory. In the Hamiltonian formalism the partition function of CS theory, after eliminating gauge freedom, is precisely the partition function of the Wess-Zumino-Witten topological quantum mechanics when the Kähler manifold M^{2n} is the moduli space of flat connections of a G-bundle over a Riemann surface Σ. Moreover, the inclusion of Wilson lines in the theory modifies the constraint equation, leading to WZW quantum mechanics over the direct product of the moduli space with n co-adjoint orbits G/T corresponding to irreducible representations of G. In this way the braid invariants previously alluded to are related to the Witten-Jones invariants of Reference [1].

The pawn moves on the chessboard of Figure 1 are now nearly complete. From d to $d+1$ the move is stochastic quantization; from d to $d+2$ it is the coupling provided by restricting the $(d+2)$-dimensional base (space) manifold to a d-dimensional submanifold. Implicitely we are linking WZW topological quantum mechanics with CS topological quantum field theory, $(0+1)$ versus $(2+1)$-dimensions, and the supersymmetric sigma model with supersymmetric Yang-Mills Theory, $(1+1)$ versus $(3+1)$-dimensions. They are very different: the systems in $d=2$ or 3 dimensions are not yet rigorously defined, while the supersymmetric sigma model has been constructed in [11] and quantum mechanics presents no problems of divergences at all. How is it possible that they yield similar topological invariants? The answer is that the topological obstructions, being independent of the parameters, are essentially captured in the semiclassical limit where the divergences of quantum field theories in $d=3$ or 3 are harmless.

5. On the physical side, the supersymmetric sigma model describes one-dimensional condensed matter systems with a symplectic manifold as the space of states of a tensorial order parameter, the order parameter being constrainded to be on Lagrangian submanifolds in the boundary of the material. The structure of the ground state, corresponding to the breaking of symmetries of the finite groups, is of the kind arising in blue phases [12] of crystalline liquids. The model also displays the phenomenon of fermion fractionization [13], due to the fact that the Dirac sea, defined at different ground states, is different and essentially tied to Floer homology. The expectation value of the vertex operators at the global ground state tells us that the model describes a plasma phase, with a mass gap and dipole dissociation [14], at least at low temperature.

The WZW topological quantum mechanics can be interpreted physically as a generalization of the Polyakov topological theory of spin to $(2n+1)$-dimensional and curved universes [15]. The propagator of a relativistic particle moving in $(2n+1)$-dimensional universes and the velocities defined on a Kähler manifold M, with spin factors given by the partition function of the WZW topological quantum mechanics, is that corresponding to the Dirac operator in a $(2n+1)$-dimensional curved space. It seems that topological quantum mechanics is half-way between the index theorem and the Dirac operator!

The organization of the paper is a follows: in Section §2 we introduce the supersymmetric sigma model and describe Floer homology, in Section §3 we deal with topological quantum mechanics and knot invariants. Finally, in Section §4 we explain the stochastic quantization procedure and discuss the physical applications, and Section §5 is devoted to work out an exact example.

2. THE MODEL AND FLOER HOMOLOGY

In the space of maps Ω from the unit interval I to a symplectic manifold M with endpoints in intersecting Lagrangian submanifolds we shall envisage the exterior algebra as the forms formally realized as fermionic quantum operators, the exterior derivative

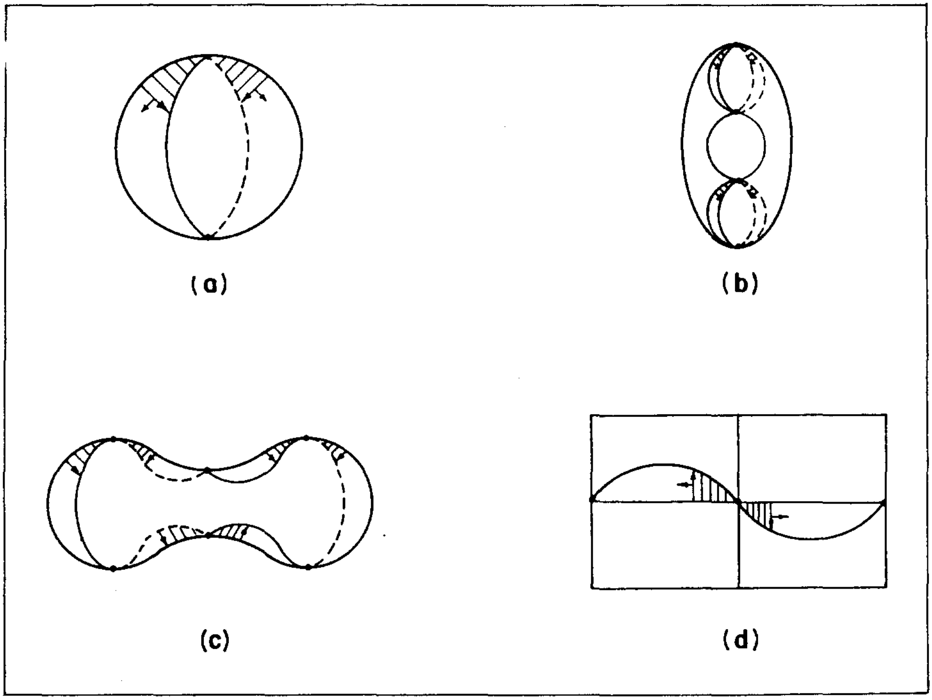

Fig. 2. Holomorphic curves, instantons, in some two-dimensional Kähler manifolds bounded by intersecting lagrangian submanifolds; a) Sphere. b) Torus. c) Hourglass. d) Cylinder.

being a (non-relativistic) Dirac-Ramond operator [16]. One can define from it, closing one's eyes to the divergences arising in infinite systems, a Laplacian operator and look for the «harmonic» forms in order to learn something of the topology of Ω .

Even in finite dimensions this is a difficult probelm to deal with but, conjugating the exterior derivative operator by the exponential of the area subtended by a loop, the deformed Laplacian is the Hamiltonian of the supersymmetric sigma model, whose divergences are fairly well controlled [11], and the harmonic forms are the ground states. In the classical limit they are wave functionals very peaked around the critical points of the area, the constant loops at intersecting points of Lagrangian submanifolds, but some of them are fake ground states when one takes into account quantum effects due to instantons. It happens that the ground states not removed by tunnel effect are those centred around the critical points of any function of M forced by the topology of M ; thus a Morse Theory for the set of fixed points of diffeomorphisms on M arises (see Fig. 2). Furthermore, the ground states reflects topological properties of Ω itself, although very subtle ones. Because the Hessian operator of the area is unbounded below all the critical points have an infinite Morse index; the important concept becomes relative Morse indices. Then Floer homology of M is a homology of the middle dimension in Ω [17]

and because the Hessian is the energy operator for the fermions the physical disguise of these concepts is the Dirac sea defined on each of the ground states.

The aim of this Section is to describe all this in detail.

2.1 The supersymmetric sigma model

Let M be a compact complex manifold of real dimension $2n$. In a local trivialization of the tangent bundle TM, where a vector field is described by its components $v^i, i = 1,\ldots,2n$, the almost complex structure, a section J of End TM such that $J^2 = -I$, corresponds to a tensor field J^i_j obeying

$$J^i_j J^j_k = -\delta^i_k . \tag{2.1}$$

We consider M equipped with a Riemannian metric g_{ij} of type (1,1); then

$$g_{ij} = J^\ell_i J^k_j g_{\ell k} \tag{2.2}$$

or $J_{ij} = -J_{ij}$ where $J_{ij} = g_{ik} J^k_\ell$. We will restrict ourselves to the case of Kähler manifolds, where the almost complex structure is parallel with respect to the Levi-Civita connection

$$D_k J^i_j = \partial_k J^i_j + \Gamma^i_{k\ell} J^\ell_j - \Gamma^\ell_{kj} J^i_\ell = 0 \tag{2.3}$$

because formulae are simpler, the general case of a symplectic manifold being conceptually equivalent.

The configuration space of the model is

$$\Omega(L_0, L_1) = \{\phi(x) \in C^\infty([0,1], M)/\phi(0) \in L_0 \text{ and } \phi(1) \in L_1\} ,$$

the space of smooth maps from the unit interval to M with endpoints in intersecting Lagrangian submanifolds, Fig. 2. In a local system of coordinates $u^i, i = 1,\ldots,2n$, elements of $T^*_\phi\Omega$, the cotangent space to Ω at ϕ, are «one-forms» of the type

$$\omega_1[\phi] = \int_0^1 \mathrm{d}x f_i[\phi, x]\delta u^i(x) \tag{2.4}$$

where by $\delta u^i(x)$ we mean the variation of an element of the basis of the cotangent bundle to M alogn ϕ with appropriate boundary conditions on L_0 and L_1. A Riemannian metric on Ω is given by the inner product of the $L^2([0,1])$ type

$$\langle\delta\phi, \delta\phi\rangle = \int_0^1 \mathrm{d}x g_{ij}(\phi)\delta u^i \delta u^i \tag{2.5}$$

meanwhile, one constructs the «exterior algebra» $\Lambda^*(\Omega)$ of Ω by means of the exterior product

$$\omega_k[\phi] = \int_0^1 \mathrm{d}x_1 \ldots \int_0^1 \mathrm{d}x_k\, f_{i_1 \ldots i_k} [\phi; x_1, \ldots, x_k] \delta u^{i_1}(x_1) \wedge \ldots \wedge \delta u^{i_k}(x) \tag{2.6}$$

where $\delta u^i(x) \wedge \delta u^j(y) = -\delta u^j(y) \wedge \delta u^i(x)$ and the $f_{ij} \ldots$ are anti-symmetric tensors contracted with the wedge product of the $\delta u' s$. In particular a «symplectic two-form» is defined by

$$W = (\delta\phi, \delta\phi) = \int_0^1 \mathrm{d}x J_{ij}(\phi) \delta u^i \wedge \delta u^j \, . \tag{2.7}$$

A key ingredient in our discussion is the definition of the «exterior differential operator», $\delta : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$, mapping k-forms in $(k+1)$-forms,

$$\delta = \int_0^1 \mathrm{d}x \frac{\delta}{\delta u^i(x)} \cdot \psi^i(x) \tag{2.8}$$

where $\psi^i(x) \equiv \delta u^i(x) \wedge$, and its adjoint $\delta^* : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$,

$$\delta^* = \int_0^1 \mathrm{d}x \frac{\delta}{\delta u^i(x)} \cdot \chi^i(x) \tag{2.9}$$

with $\chi^i(x) \equiv \imath(\delta u^i(x))$. The physicist notation for the operators $\psi^i(x)$ and $\chi^i(x)$ is justified by noticing that they satisfy anticommutation relations of the type

$$\begin{aligned} \{\chi^i(x), \psi^j(y)\} &= g^{ij}(\phi)\delta(x-y) \\ \{\chi^i(x), \chi^j(y)\} &= \{\psi^i(x), \psi^j(y)\} = 0 \, , \end{aligned} \tag{2.10}$$

i.e. they are realized by fermionic quantum operators.

The formal Laplacian $\Delta = \delta^*\delta + \delta\delta^*$ acting on a functional $F[\phi] : \Omega \to \mathbb{R}$ which is C^2 and L^2 in the functional sense can be regularized in the form

$$\Delta F[\phi] = \lim_{s \to 0} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \frac{\delta^2 F}{\delta u^i(x) \delta u^j(y)} g_{ij}(\phi) G_S(x-y) \tag{2.11}$$

where $G_S(x-y)$ is the kernel of $-\frac{\mathrm{d}^2}{\mathrm{d}^2 x}$. At this point, «infinite dimensional» Hodge theory may be considered. One could try to describe the topology of Ω in terms of harmonic «functional» forms of Δ. The problem, being too difficult to be managed by analytic methods, can be approached in an indirect way. Indeed Δ can be thought of as a supersymmeytric Hamiltonian with respect to the symmetry generated by δ. The following relations

$$H_0 = \{\delta^*, \delta\} \quad (\mathbf{a}) \quad \delta^2 = \delta^{*2} = 0 \quad (\mathbf{b}) \quad [\delta, H_0] = 0 \quad (\mathbf{c}) \tag{2.12}$$

define a supersymmetric system for which the ground states $H_0\psi_k[\phi] = 0$ are the harmonic functional k-forms of Δ. The idea is to look for a Hamiltonian, some deformation of H_0, such that the ground states can be easily found in some special limit while the «Betti numbers» coming from the new Laplacian are the same as those coming from Δ.

Following Floer we extend our configurations space to be

$$\Omega_\tau(L_0, L_1) = \{\phi(x,\tau) \in C^\infty([0,1] \times \mathbf{R}, M)/\ \phi(0,\tau) \in L_0 \text{ and } \phi(1,\tau) \in L_1\},$$

introducing an extra parameter τ («euclidean» time), and considering the vector fields belonging to $T_\phi\Omega_\tau$

$$\dot\phi(x,\tau) = \sum_{i=1}^{2n} \frac{\partial u^i}{\partial x} \cdot \frac{\delta}{\delta u^i} \quad \phi'(x,\tau) = \sum_{i=1}^{2n} \frac{\partial u^i}{\partial \tau} \cdot \frac{\delta}{\delta u^i}, \tag{2.13}$$

respectively in the directions of $[0,1]$ and $\mathbf{R}$; the Wess-Zumino-Witten functional [18] is

$$\begin{aligned} \mathcal{A}[\phi] &= W(\dot\phi, \phi') = \int_\varepsilon \omega = \int_\varepsilon J_{ij}(\phi) \mathrm{d}u^i \wedge du^j \\ &= \int_{-\infty}^{\infty} \mathrm{d}\tau \int_0^1 \mathrm{d}x J_{ij}(\phi) \left(\frac{\partial u^i}{\partial x}\frac{\partial u^j}{\partial \tau} - \frac{\partial u^i}{\partial \tau}\frac{\partial u^j}{\partial x} \right), \end{aligned} \tag{2.14}$$

i.e. the area bounded by $\phi(x,-\infty)$, $\phi(x,\infty)$, L_0 and L_1. Stoke's theorem tell us that: 1) $\mathcal{A}$ is in fact a functional defined on Ω, not in Ω_τ. 2) $\mathcal{A}$ is a topological invariant, independent of the choice of L_0 and L_1. 3) It is ambigous: $\mathcal{A}$ changes by $4\pi k$ by covering k-times the fundamental two-cycle in $H_2(M;\mathbf{R})$.

Define new differential operators by

$$\begin{aligned} \delta_s &= e^{-s\mathcal{A}} \delta e^{s\mathcal{A}} = \delta + s\delta\mathcal{A} \\ \delta_s^* &= e^{s\mathcal{A}} \delta^* e^{-s\mathcal{A}} = \delta^* - s\delta^*\mathcal{A}. \end{aligned} \tag{2.15}$$

From the gradient of $\mathcal{A}$

$$\delta\mathcal{A} = \int_0^1 \mathrm{d}x \left\{ J^i_j[\phi] \frac{\mathrm{d}u^j}{\mathrm{d}x} \right\} \psi_i(x) \tag{2.16}$$

one finds easily that δ_s,

$$\delta_s = \int_0^1 \mathrm{d}x \left\{ \frac{\delta}{\delta u^i} + sJ_{ij}[\phi] \frac{\mathrm{d}u^j}{\mathrm{d}x} \right\} \psi^i(x) , \tag{2.17}$$

is an operator of the Dirac-Ramond type with a second term coming from the vector field $J\frac{\mathrm{d}}{\mathrm{d}x}$, orthogonal to the loop, instead of the generator of loop rotations. The deformed Laplacian is still a supersymmetric Hamiltonian, with supersymmetry generator δ_s,

$$H_s = \{\delta_s^*, \delta_s\} = \{\delta^*, \delta\} + s^2(\delta\mathcal{A})^2 + s \int_0^1 \mathrm{d}x \psi^i(x) \frac{\delta^2 \mathcal{A}}{\delta u^i \delta u^j} \chi^j(x) ;$$

explicitely, it is

$$\begin{aligned} H_s = & H_0 + s^2 \int_0^1 \mathrm{d}x g_{ij}[\phi] \frac{\mathrm{d}u^i}{\mathrm{d}x} \frac{\mathrm{d}u^j}{\mathrm{d}x} \\ & + s \int_0^1 \mathrm{d}x \phi^i(x) J_{ij}[\phi] \frac{\mathrm{d}\chi^j}{\mathrm{d}x} \\ = & \int_0^1 \mathrm{d}x \left\{ -g^{ij}(\phi) \frac{\delta^2}{\delta u^i \delta u^j} + s^2 g_{ij}[\phi] \frac{\mathrm{d}u^i}{\mathrm{d}x} \frac{\mathrm{d}u^j}{\mathrm{d}x} \right. \\ & \left. + s\psi^i J_{ij}[\phi] \frac{\mathrm{d}\chi^j}{\mathrm{d}x} \right\} . \end{aligned} \tag{2.18}$$

The key idea is that dim Ker δ = dim Ker δ_s and dim Im δ = dim Im δ_s because functional in the kernel of δ are in one-to-one correspondance with those in the kernel of δ_s just by undoing the conjugation by $e^{s\mathcal{A}}$.

2.2. Local ground states

The ground states of H_s could therefore provide a model for the homology of Ω and it happens that in the limit $s \to \infty$, the classical limit in physical terminology, they are very peaked gaussians centred around the points where $\delta\mathcal{A} = 0$, the critical points of $\mathcal{A}$. These are the constant loops, the points where L_0 and L_1 meet, which are non-degenerate if the intersections are transversal. This suggests Morse Theory but a very peculiar one because the spectrum of the Hessian

$$\delta^2 \mathcal{A}|_{\phi_c} = \int_0^1 \mathrm{d}x J_{ij}[\phi_c] \frac{\mathrm{d}}{\mathrm{d}x} \tag{2.19}$$

is unbounded below; this means that all the critical points have a Morse index $\mu(\phi_c)$, the dimension of the negative eigenspace of $\delta^2\mathcal{A}|_{\phi_c}$, equal to infinity. One cannot try to relate in the traditional way the critical points of $\mathcal{A}$ with the topology of Ω and, to cope with this situation, we define «regularized» or «relative» Morse indices $\mu_R[\phi_c] = \mu[\phi_c] - \mu[\phi_0]$ with respect to one critical point ϕ_0 selected as a reference. We shall deal with «middle dimensional» homology, a concept explained in the fourth paper of Reference [2] and to be developed later in this paper.

The Hamiltonian near a critical point ϕ_c is obtained by expanding the «bosonic» and «fermionic» fields in terms of eigenfunctions of the Hessian

$$
\begin{aligned}
&J^i_j[\phi_c]\frac{df^j_n}{dx} = \lambda_n f^i_n \\
&u^i(x) = u^i_c + \sum_n a_n f^i_n(x) \; ; \\
&\psi^i(x) = \sum_n c_n f^i_n(x) \; ; \; \chi^i(x) = \sum_n c^*_n f^i_n(x) \\
&\{c^*_n, C_m\} = \delta_{nm} \; ; \; \{c^*_n, c^*_m\} = \{c_n, c_m\} = 0 \; .
\end{aligned}
\tag{2.20}
$$

This is tantamount to using Morse coordinates [19]; the spectrum of H_s fulfil $Sp\, H_s \simeq \oplus_{\text{Crit}\, A} Sp\, H^c_s$ when $s \to \infty$ and H^c_s is

$$
H_s = \sum_n \left[-\frac{\partial^2}{\partial a_n^2} + s^2 \lambda_n a_n^2 + s\lambda_n [c^*_n, c_n] \right] . \tag{2.21}
$$

The spectrum is

$$
Sp\, H^c_s = \sum_n \{(2p_n + 1)|\lambda_n| + (2q_n - 1)\lambda_n\} \; , \tag{2.22}
$$

where $p_n = 0, 1, 2, 3, \ldots$ and $q_n = 0$ or 1, because H^c_s is a sum of harmonic oscillators Hamiltonians both of the bosonic and fermionic type [20]. The ground states are therefore harmonic oscillator ground states, $p_n = 0$ and gaussian wave functions around ϕ_c, with fermionic occupation number $q_n = 1$ or zero according to whether λ_n is negative or positive. The explicit analytic form for the local ground states, renormalized with respect to the trivial case, is

$$
\begin{aligned}
&\psi^c \left[a, c^-_1, c^-_2, \ldots, c^-_{\mu_R(\phi_c)} \right] = \\
&= \frac{\operatorname{Det}^{1/2} \left| J_{ij}(\phi_c)\frac{d}{dx} \right| e^{-s\sum_n \lambda_n a_n^2/2}}{\operatorname{Det}^{1/2} \left| J_{ij}(\phi_0)\frac{d}{dx} \right| e^{-s\sum_n \lambda^0_n (a^0_n)^2/2}} \cdot c^-_1 \wedge c^-_2 \wedge \ldots \wedge c^-_{\mu_R}
\end{aligned}
\tag{2.23}
$$

where the c^-_n are the «fermionic» Fourier coefficients corresponding to the eigenfunctions f^i_n of negative λ_n that in reference case $(f^0_n)^i$, were of λ^0_n positive. Expression

(2.23) allows one to understand what middle dimensional cohomology means: for any ϕ_c there is an infinite number of both c_n^+ and c_n^- ! The cycles related with the critical points ϕ_c have a dimension such that there is both an infinite number of cycles of greater and lower dimension!

At this point we have proved a kind of weak Morse inequality: the number of critical points of Morse index $\mu_R = p$ is greater than or equal to the dimension opf the space subtended by the cocycles of order p in the middle dimensional cohomology. Equality is attained if the ground states previously found are not removed by quantum corrections, they remain as zero energy eigenfunctionals even when the crude large s approximation, $H_s = \oplus_{\mathrm{Crit}\ \mathcal{A}} H_s^c$ is not reliable.

2.3. Instanton homology

There is one important drawback: our derivation is intrinsically local and deals with the critical points as isolated. Global analysis of the problem requires one to take into account links between different critical points coming from tunnel effect on the physical side and leading to the strong Morse inequalities on the topological one. For a «path» $\phi(x,\tau)$ such that $\lim_{\tau\to-\infty}\phi(x,\tau) = \phi_{c_1}$ and $\lim_{\tau\to\infty}(\phi(x,\tau) = \phi_{c_2}$ we consider the family of Hessians

$$h(\tau) = \delta^2 \mathcal{A}|_{\phi(x,\tau)} = \int_0^1 \mathrm{d}x J\left[\phi(x,\tau)\right] \frac{\mathrm{d}}{\mathrm{d}x} .$$

The spectral flow of $h(\tau)$ is a topological invariant which can be shown to be

$$SF\ h(\tau) = \mu\left[\phi_{c_2}\right] - \mu\left[\phi_{c_1}\right] = \mu_R\left[\phi_{c_2}\right] - \mu_R\left[\phi_{c_1}\right] \tag{2.24}$$

by computing the index of the linearization of the operator $\bar{\partial}_J = \frac{\partial}{\partial\tau} + J\frac{\partial}{\partial x}$ to $\bar{\partial} = \frac{\partial}{\partial\tau} + \frac{\partial}{\partial x} + N$ (in local coordinates). With appropriate boundary conditions and deforming N to zero, it is possible to show that $\bar{\partial}$ is Fredholm, its index amounts to $SF\ h(\tau)$ and applying the Riemann-Roch theorem one arrives at formula (2.24) (see Ref. [3]). Because Ω is not simply-connected, essentially $H_1(\Omega;\mathbf{R}) \simeq H_2(M;\mathbf{R})$, the spectral flows of $h(\tau)$ is not well defined; it jumps by $2N$, if the first Chern class of M properly normalized is $C_1(M) = N[w]$ and $[w]$ is the class of w in $H^2(M)$, when the path between ϕ_{c_1} and ϕ_{c_2} is non-contractible.

In physical terms spectral flow is reminescent of tunnel effect by instantons [21]. The interesting quantity is

$$f_\beta(b,a) = \langle\phi_{c_b}^{(p+1)}(\beta)|\delta\phi_{c_a}^{(p)}(0)\rangle = \langle\phi_{c_b}^{(p+1)}|\delta e^{-\beta H_s}|\phi_{c_a}^{(p)}\rangle , \tag{2.25 a}$$

the amplitude of transition from the ground state around ϕ_{c_a} with p fermions, a p-form , at $\tau = 0$ to the ground state around ϕ_{c_b} with $p+1$ fermions at $\tau = \beta$; notice tat the

δ operator acting on a p-form gives a $(p+1)$-form. Another expression for $f_\beta(b,a)$ is provided by path integral quantization of supersymmetric theories [22],

$$f_\beta(b,a) = \int_{\Omega_r} [\mathrm{d}\, u^i] \int_{T^*\Omega_r} [\mathrm{d}\, \eta^i][\mathrm{d}\, \xi^i] \rho e^{-S_\beta} \tag{2.25 b}$$

with the boundary conditions $\mu^i(x,0) = u^i_{c_1}$, $u^i(x,\beta) = u^i_{c_2}$ and $\xi_i(x,0) = \xi_i(x,\beta) = a_i$. Here S_β is the «euclidean» action

$$\begin{aligned} S_\beta = \int_0^\beta \mathrm{d}\,\tau \int_0^1 \mathrm{d}\,x \Big\{ & g_{ij} \left(\frac{\partial u^i}{\partial \tau} \cdot \frac{\partial u^j}{\partial \tau} + s^2 \frac{\partial u^i}{\partial x} \cdot \frac{\partial u^j}{\partial x} \right) + \\ & + \xi^i g_{ij} \left(\frac{D}{\partial \tau} + s J_{ij} \frac{\partial}{\partial x} \right) \eta^i + \frac{1}{4} R_{ijk\ell} \eta^i \xi^j \eta^k \xi^\ell \Big\} \\ \frac{D\eta^i}{\partial \tau} = \frac{\partial \eta^i}{\partial \tau} + \Gamma^i_{k\ell} & \frac{\partial u^k}{\partial \tau} \eta^\ell \end{aligned} \tag{2.26}$$

which upon quantization in the Schrödinger representation gives rise to the Hamiltonian H_s; η_i, ξ^i are the Grassman variables

$$(\eta^i)^2 = (\xi^i)^2 = 0, \quad \left\{ \eta^i(x,\tau), \xi^j(y,\tau) \right\} = 0$$

which are realized at the quantum level as the operators χ^i, ψ_i and $\rho = \int_0^1 \mathrm{d}\, x g_{ij} \frac{\partial u^i}{\partial \tau} \xi^i$ is the classical counterpart of the δ operator.

At this point we pause to make a comment: the functional integrals formally written in (2.25 b) have been rigorously defined and proved to be equals to the amplitude (2.25 a) in Reference [11]. They are of the same character than those functional integrals which appear in the field theoretical proofs of index theorems for the Dirac operator in Loop Space [23].

In the limit $\beta \to \infty$ (2.25 a) yields a formula

$$\begin{aligned} f_\beta(b,a) &= \sum_n \psi_n^{(p+1)*} \left[\phi_{c_b}\right] \delta \psi_n^{(p)} \left[\phi_{c_a}\right] e^{-\beta\lambda_n} \overset{\beta\to\infty}{\simeq} \\ & \psi_0^{(p+1)*} \left[\phi_{c_b}\right] \delta \psi_0^{(p)} \left[\phi_{c_a}\right] e^{-\beta\lambda_0} \end{aligned} \tag{2.27}$$

connecting $f_\beta(b,a)$ with the smallest eigenvalue λ_0 of $H_s^{c_\bullet}$. On the other hand (2.25

b) can be computed, for s large, by the steepest descent method yielding

$$f_\beta^{(1)}(b,a) \overset{s\to\infty}{\simeq} \sum_{I\in\hat{M}(\phi_{c_b},\phi_{c_a})} \left\{\int_0^1 \mathrm{d}\,x J_{ij}\,[\phi_I]\,\frac{\partial u_I^i}{\partial x} a^j\right\} e^{-s|A(\phi_{c_b})-A(\phi_{c_a})|} \cdot \beta\gamma(I)\frac{\mathrm{Pfaff}'\Delta_F}{\mathrm{Det}^{1/2}\Delta_B} \tag{2.28}$$

$$\Delta_{F_{ij}}(\phi_I) = i g_{ij}^I \frac{D^I}{\partial \tau} + s J_{ij}^I \frac{\partial}{\partial x}\,;$$

$$\Delta_{B_{ij}}(\phi_I) = -\,g_{ij}^I\left(\frac{D^I}{\partial\tau}\right)^2 - s^2 g_{ij}^I \frac{\partial^2}{\partial x^2}\,.$$

To derive (2.28) the order of the arguments is as follows:

1. The main contribution to the functional integral (2.25 b) comes, when $s\to\infty$, from the absolute minima of S_β. In the «fermionic» sector they are very symple: $\xi_i^I(x,\tau) = a_i$ and $\eta_i^I(x,\tau) = 0$. For the «bosonic» variables they are the solutions of the flow induced by the gradient $\delta\mathcal{A}$

$$\bar\partial_J \phi^I = \frac{\partial\phi^I}{\partial\tau}(x,\tau) + J\left[\phi^I(x,\tau)\right]\frac{\partial\phi^I}{\partial x}(x,\tau) = 0 \tag{2.29}$$

with initial and final conditions $\phi(x,-\infty) = \phi_{c_a}$, $\phi(x,\infty) = \phi_{c_b}$. What physicists call instantons are in this case J-holomorphic maps from $\mathbb{R}\times I \subset \mathbb{C}$ to M. A compactness condition for the set of such maps (see Ref. [3]) can be shown by considering the Banach space structure induced by the area and its dimension computed by the index theorem applied to the linearization of $\bar\partial_J$ on ϕ^I. Translations in $\mathbb{R}$ act on this «moduli space» of solutions $M(\phi_{c_b},\phi_{c_a})$ of (2.29) and the index theorem tells us that

$$\dim\hat{M}\left(\phi_{c_b},\phi_{c_a}\right) = \mu\left[\phi_{c_b}\right] - \mu\left[\phi_{c_a}\right] - 1 \tag{2.30}$$

where $\hat{M} = M/\mathbb{R}$; for critical points with Morse indices differeing by one unit $\hat{M}$ is a finite set.

2. Berezin integration on the small deformations of the fermionic variables leaves us with the Gaussian bosonic measure

$$\mathrm{d}\,\mu_a \overset{s\to\infty}{\simeq} \mathrm{Pfaff}'\Delta_F(\phi_I)\,\left[\mathrm{d}\,\vec\phi\right] e^{-s|A(\phi_{c_b}-A(\phi_{c_a})|}\cdot e^{-\langle\vec\phi', J_I\dot{\vec\phi}\rangle} \tag{2.31}$$

which can be integrated to give $\beta\mathrm{Det}^{-1/2}\Delta_B\cdot\mathrm{Pfaff}'\Delta_F\cdot e^{-s|A(\phi_{c_b}-A(\phi_{c_a})|}$.

3. Because of supersymmetry the quotient of the primed determinants (excluding the zero eigenvalues due to translations in $\mathbf{R}$) is one (1) if $\mu(\phi_{c_b}) - \mu(\phi_c) = 1$ and zero otherwise.

4. $\gamma(I) = \pm 1$ according to how the orientation of the tangent space to M at ϕ_{c_b} orthogonal to the vector tangent to the holomorphic surface in the $\frac{\partial \phi_I}{\partial \tau}$ direction is, compared with the tangent space to M at ϕ_{c_a}.

5. In the approximation where the contribution to (2.25 a) of n instantons is n times the contribution of one instanton, the Dilute Gas Approximation,

$$f_\beta(b,a) = \sum_n \left(f_\beta^{(1)}(b,a)\right)^n, \quad n = 0,1,2,\ldots$$

and

$$\lambda_0 = \sum_{I \in \hat{M}} \gamma(I) e^{-s|A(\phi_{c_b} - A(\phi_{c_a})|}. \tag{2.32}$$

The combination of signs in (2.32) which makes $\lambda_0 = 0$ considering the instanton (or tunneling) contribution appear precisely for the critical points forced by the topology of M. Also, the states constructed in (2.23) around ϕ_c are no longer ground states when semi-classical corrections are taken into account if ϕ_c correspond to a critical point which is not related to the topology of M. The middle dimensional cohomology of Ω is tied to the ground states not removed by tunnel effects.

Floer makes this more precise, and mathematically sound, by considering the following Morse complex: in the free module generated by the critical points of the area he defines a co-boundary operator $\partial : C^p \to C^{p+1}$ by $\partial|a >= n(b,a)|a >$ where $n(b,a)$ is

$$n(b,a) = \langle b|\partial|a\rangle = \frac{e^{+s|A(\phi_{c_b} - A(\phi_{c_a})|}}{\beta\langle a, J\dot{\phi}_I\rangle} \cdot f_\beta^{(1)}(b,a). \tag{2.33}$$

Obvioulsy $\partial^2 = 0$ and the Floer cohomology groups

$$HF^p(M;\mathbf{R}) = \frac{\ker \partial|_{c_p}}{\operatorname{Im} \partial|_{c_{p-1}}},$$

defined modulo $2N$, are related on one hand to the middle dimensional homology of Ω and, on the other hand, to the topology of M.

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(1) Provided the spectral flow of $J\,[\phi_I]\,\frac{d}{dx}$ is not taken into account.

3. TOPOLOGICAL QUANTUM MECHANICS

3.1. Global ground states

In any supersymmetric model, where the interactions are constructed from a superpotential [24], a zero energy eigenstate is immediately found by inspection. In our case is easy to see that

$$\psi_0[\phi] = e^{-s\mathcal{A}[\phi]} = e^{\int_{-\infty}^{\infty} \mathrm{d}\tau \int_0^1 \mathrm{d}x J_{ij}(\phi)\left(\frac{\partial \psi^i}{\partial \tau}\cdot\frac{\partial \psi^j}{\partial x} - \frac{\partial \psi^i}{\partial x}\cdot\frac{\partial \psi^j}{\partial \tau}\right)} \tag{3.1}$$

is a «zero-form» eigenfunctional of H_s with zero energy

$$H_s e^{-s\mathcal{A}(\phi)} = 0$$

because

$$\begin{aligned} \delta_S \psi_0[\phi] &= \delta e^{-s\mathcal{A}(\phi)} + s\delta\mathcal{A}(\phi) e^{-s\mathcal{A}(\phi)} = 0 \\ \delta_s^* \psi_0[\phi] &= 0 \end{aligned} \tag{3.2}$$

independently of any approximation!

The reason for this is the following: suppose we start with the spectral equation

$$H_0 \psi_E[\phi] = E\psi_E[\phi] \,. \tag{3.3}$$

The wave functional $\tilde{\psi}_E[\phi] = e^{-i\alpha F[\phi]}\psi_E[\phi]$, where $F : \Omega \to \mathbf{R}$, satisfies a similar equation with δ replaced by the «covariant derivative» $\delta_\alpha = \delta + i\alpha\delta F$

$$H_\alpha \tilde{\psi}_E[\phi] = \{\delta_\alpha^*, \delta_\alpha\}\, \tilde{\psi}_E[\phi] = E\tilde{\psi}_E[\phi] \,. \tag{3.4}$$

Our model is a particular case with a «complex phase», $i\alpha = s$ and F the area functional, of this situation

$$H_s \tilde{\psi}_E[\phi] = E\tilde{\psi}[\phi] \; ; \; \tilde{\psi}_E[\phi] = e^{-s\mathcal{A}(\phi)}\psi_E[\phi] \,. \tag{3.5}$$

This means that the dimension of the harmonic eigenspaces of Δ and Δ_s are the same; also there are states of this kind of any order, with any number of fermions. In the order zero case $\psi_0[\phi]$ = constant and $\tilde{\psi}_0[\phi] = e^{-s\mathcal{A}[\phi]}$ are linked by a complex gauge transformation.

This kind of zero energy state, the exponential of the integral of the superpotential, is very important in supersymmetry. If they are normalizable, they are the ground state of the model and supersymmetry is unbroken [25]. The next step is thus the study of the norm of $\tilde{\psi}_0$ in our model: if the norm is a real number, $\tilde{\psi}_0$ would be the exact ground state and the discussion of the previous Section would be superfluous! We will see, however that $\psi_0[\phi] = e^{-sA(\phi)}$ is not a good quantum state, it is not in $L^2(\Omega)$, and in the analysis of the divergences of the norm of ψ_0 the Floer homology will reappear.

The norm is the functional integral

$$N(s) = \int_\Omega [d\phi]\, e^{-2sA(\phi)}\ , \tag{3.6}$$

a Wiener-like integral which, for s large, can be approached by the Steepest Descent Method,

$$\begin{aligned} N(s) &\overset{s\to\infty}{\simeq} \sum_{\text{Crit}\,A} I(s,\phi_c)\, e^{-2sA[\phi_c]} \\ I(\phi) &= \int_{T\Omega|\phi_c} \\ &[\mathrm{d}v^i]\, e^{-\int_0^1 \mathrm{d}x \left\{ v^i(x) J_{ij}[\phi_c] \frac{\mathrm{d}v^j}{\mathrm{d}x} \right\}}\ ;\ v^i(x)=u^i(x)-u^i_c, v^i(0)\in TL_0, v^i(1)=TL_1\ . \end{aligned} \tag{3.7}$$

The quadratic form in the Gaussian integral in (3.7) is degenerate due to the contribution of constant vector fields and a Fadeev-Popov measure must be used to remove this spurious integration volume. To understand which symmetry is at stake, notice that the «holonomy» operator

$$R = e^{i\int \mathrm{d}x \Gamma^i_{jk}(\phi) \frac{\mathrm{d}u^k}{\mathrm{d}x} \omega^j_i}$$

acts by «rotating» the whole M, L_0, L_1 system in a chart near the intersection point via the $O(2n)$-connection $A^i_j(\phi) = \Gamma^i_{jk}(\phi)\frac{\mathrm{d}u^k}{\mathrm{d}x}$ with «rotation» angles ω^j_i. Near ϕ_c it is convenient to use a normal system of coordinates and, for displacements induced by constant vector fields, we have that:

$$\begin{aligned} u^i(x) &= m^i + xv^i\ , \quad \Gamma^i_{jk}(\phi) = R^i_{jk\ell}(\phi_c)\, x v^\ell \\ A^i_j(\phi_c) &= \int_0^1 \mathrm{d}x A^i_j(\phi) = \frac{1}{2} R^i_{jk\ell}(\phi_c) v^k v^\ell\ . \end{aligned} \tag{3.8}$$

We are implicitely assuming that the symplectic manifold M^{2n} is a homogeneus one, locally isomorphic to a co-adjoint orbit of a central extension by $\mathbb{R}$ of the $2n$-dimensional translation group: the Heisenberg-Weyl group. Let us recall, [26], that the transformations in (3.8) can be seen as the representation of the Lie algebra of $Sp(2n)$ in the set of polynomials of degree two in the v^i variables governed by the relations $\{v^i, v^j\} = J^{ij}[\phi_c]$ and with Weyl ordering given by the coefficients $c_{k\ell} = R^i_{jk\ell}(\phi_c)\omega^j_i$.

Let T be the representation of the Lie algebra of the Heisenberg group in the set of polynomials of degree one in the Weyl algebra, generating translations in M^{2n}, given by

$$T = \left(\frac{\partial}{\partial m^i} + \Gamma^k_{ij}(\phi)\,\mathrm{d}\,u_k \mathrm{d}\,u^j\right) v^i .$$

Then,

$$(3.9)\qquad \begin{aligned} T^+ \otimes T &= v_i A^i_j(\phi_c) v^j \\ A^i_j(\phi_c) &= R^i_{jk\ell}(\phi_c)\,\mathrm{d}\,u^k \mathrm{d}\,u^\ell \end{aligned}$$

and the transformation R, preserving the complex structure, is symplectic. Locally, M^{2n} is of the form $Sp(2n)/T^n$, the transformations in the maximal torus leaving invariant the intersection point, Fig. 3.

The symmetry related to the kernl of the quadratic form appearing in the SPA to the path integral is thus the symmetry under «rigid» $O(2n)$ transformations preserving the complex structure. The formula for I is the following, see [27]

$$(3.10)\qquad I[\phi_c] = \left.\frac{\mathrm{Det}^{1/2} JT^+ \otimes T}{\mathrm{Det}^{1/2}\left|J\left(\frac{d}{dx} + T^+ \otimes T\right)\right|}\right|_{\phi_c} \cdot e^{i\pi/2\mu(\phi_c)} .$$

It is convenient to express $A(\phi_c)$ is canonical form, reducing the $O(2n)$ bundle to the

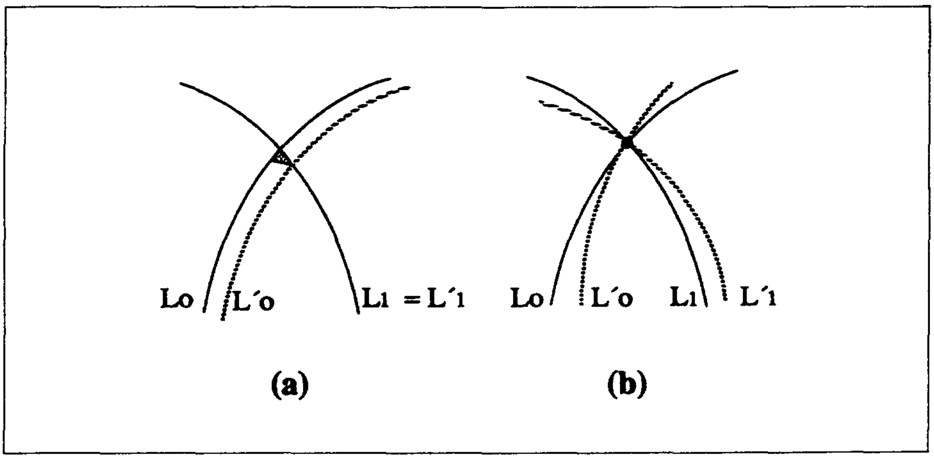

Fig. 3. Rigid transformations acting on the whole system of Lagrangian intersecting submanifolds.

normal form $E = L_1 \oplus \ldots \oplus L_n$. Then, $A(\phi_c)$ is off-diagonal by blocks

$$A(\phi_c) = \frac{1}{2}\begin{pmatrix} 0 & \alpha_1(\phi_c) & & \\ -\alpha_1(\phi_c) & \ddots & & \\ & & 0 & \alpha_n(\phi_c) \\ & & -\alpha_n(\phi_c) & 0 \end{pmatrix}$$

and we are left with the transformations not in T^n.

Regularinzing the determinant by the ξ-function method [28], writing A and J in canonical form and taking into account the Jacobian of the action of the $SO(2)$ subgroup of $O(2n)$ in the choice of normal coordinates around ϕ_c, $\det^{1/2} k = \prod_{i=1}^n e^{\pi/2\alpha_i(\phi_c)}$, the final result is

(3.12) $$I(\phi_c) = \prod_{i=1}^{n} \frac{1 - e^{-\pi\alpha_i(\phi_c)}}{2(1 - \cos h\pi\alpha_i(\phi_c))} \cdot e^{i\pi/2\mu[\phi_c]} .$$

The norm of $\psi_0[\phi] = e^{-sA[\phi]}$ is thus desperately divergent: $\mu(\phi_c) = \infty$ for any ϕ_c. But $\mu(\phi_c)$ is the dimension of the negative eigenspace of the hessian of A at ϕ_c, the energy operator for the fermions in the system: the reason for the divergences is the infinite dimension of the negative energy fermionic eigenspace, the Dirac sea. Physicists usually cope with this situation by renormalizing the vacuum energy. This amounts to replacing $\mu(\phi_c)$ by $\mu_R(\phi_c)$ and defining a «renormalized» norm,

(3.13) $$N^R(s) = \sum_{\text{Crit } A} |I(\phi_c)|\, e^{i\pi/2\mu_r(\phi_c)} e^{-2sA[\phi_c]} ,$$

with an obvious topological meaning: only in the trivial case, $M = \mathbf{R}^{2n}$ with a single critical point $\phi_c = 0$ and $\mu_R(\phi_c) = 0$, is $N^R(s)$ a real quantity. In the case of M being topologically non-trivial, $N^R(s)$ is a sum of terms with, at least, as many phase factors as critical points forced by the topology of M. To be more precise, the Floer homology is a lower bound to $N^R(s)$,

(3.14)
$$N^R(s) \geq \sum_{p=0}^{2N} \left|I\left(\phi_c^{(p)}\right)\right| e^{i\frac{\pi}{2}p} f_p \cdot e^{-2sA\left(\phi_c^{(p)}\right)};$$
$$\mu_R\left(\phi_c^{(p)}\right) = p, \quad f_p = \dim HF^p(M);$$

here the inequality is to be understood in the sense that $N^R(s)$ has at least the terms written on the right-had side of (3.14). There are more terms, those due to the critical points removed by instantons effects. This is why we called in Ref. [7] an analogous wave functional, in a supersymmetric Yang-Mills system, the global ground state. Wave functionals of the kind include all the candidates to be ground states of these models but, being non-normalizable, they are no even admisible as quantum states. We think however that the previous discussion sheds some light on the problem posed by Jackiw sometime ago [29]: we known at least why zero energy eigenfunctionals on non-linear Schrödinger functional differential equations are not normalizable. It is due to supersymmetry breaking by instantons.

3.2. Topological invariants

Let us take now another viewpoint. By considering s as imaginary we receive a model in quantum mechanics with a partition function (or quantum action)

$$\mathbf{Z}(k) = \int_\Omega [\mathrm{d}\phi] e^{ik\mathcal{A}[\phi]} \; ; \; ik = -2s, \; 2k \in \mathbf{Z} \; . \tag{3.15}$$

Note that taking k as an integer or half-integer the ambiguity in $\mathcal{A}[\phi]$ due to $H^2(M;\mathbf{R}) \neq e$ do not effect $\mathbf{Z}(k)$. The topological meaning of $\mathcal{A}[\phi]$ gives this model, living in a spatial dimension $d = 0$, the flavour of a topological field theory of the type of the Chern-Simons $(2+1)$-dimensional theory. We will call the model described by $\mathbf{Z}(k)$ in (3.15) Wess-Zumino-Witten topological quantum mechanics. To unveil its topological meaning, related with the target manifold M, we do a computation for $\mathbf{Z}(k)$ similar to that previously perfomed for $N(s)$. The Stationary Phase Approximation for $\mathbf{Z}(k)$ yields, for k large,

$$\begin{aligned} &\mathbf{Z}(k) \overset{k\to\infty}{\simeq} \sum_{\text{Crit } 0} e^{ik\mathcal{A}(\phi_c)} \\ &\frac{\mathrm{Det}^{1/2} J(\phi_c)(T^+ \otimes T)(\phi_c)}{\mathrm{Det}^{1/2} \left| J(\phi_c) \left\{ \frac{\mathrm{d}}{\mathrm{d}x} + T^+ \otimes T(\phi_c) \right\} \right|} \cdot e^{i\pi/4\, \eta_{h[\phi_c]}(0)} \; ; \end{aligned} \tag{3.16}$$

here $\eta_{h[\phi_c]}(0)$ is the spectral asymmetry of the hessian operator

$$h[\phi_c] = J(\phi_c) \left\{ i\frac{\mathrm{d}}{\mathrm{d}x} + T^+ \otimes T(\phi_c) \right\} \; ;$$

given by a similar expression to the APS theorem [30]

$$\eta_{h[\phi_c]}(0) = \frac{1}{4\pi} \int \mathrm{d}x Tr(J A(\phi_c)) = \frac{1}{2} \left[\sum_{i=1}^{n} \alpha_i(\phi_c) \right] \mathcal{A}(\phi_c) \tag{3.17}$$

as measured with respect to the reference $\eta_{h[\phi_0]}(0)$.

The final result for the partition function is (2)

$$\mathbf{Z}(k) \overset{\to\infty}{\simeq} \sum_{\text{Crit } \mathcal{A}} e^{i(k+\rho)\mathcal{A}(\phi_c)} \cdot \prod_{i=1}^{n} \frac{1 - e^{-i\pi\alpha_i(\phi_c)}}{2(1 - \cos \pi\alpha_i(\phi_c))} \tag{3.18}$$

the contribution of η being subsummed in a shift of the integer k. This phenomenon is quite well understood in physical terms in the general framework of anomalies: integers appearing in the exponential terms of (3.11) due to topological properties of the configuration space becomes real numbers and redefine physical quantities when passing from «euclidean» to real time.

More interesting are the other kind of factors in (3.18). They suggest a fixed point formula for some topological invariant of the manifold M. The obvious candidate is the Lefschetz number for the Dolbeault complex [31]

$$L_{\mathrm{Dol}}(f) = Trf^* H^{0,\,\mathrm{even}} - Trf^* H^{0,\,\mathrm{odd}} \tag{3.19}$$

where $f : M \to M$ is a holomorphic map and $f^* H^{0,\,\mathrm{even}}$ (respectively $f^* H^{0,\,\mathrm{odd}}$) is the pullback of f acting on the cohomology of the complex via the pull-back to the bundle $\Lambda^{0,\,\mathrm{even}}(M, \mathbf{R})$ (respectively $\Lambda^{0,\,\mathrm{odd}}(M, \mathbf{R})$).

The Lefschetz fixed point formula for the Lefschetz number is

$$L_{\mathrm{Dol}}(f) = \sum_{\text{fixed points}} \left. \frac{Tr\, f^*\Lambda^{0,\,\mathrm{even}} - Trf^*\Lambda^{0,\,\mathrm{odd}}}{\det(I - \mathrm{d} f(\phi_c))} \right|_{\phi_c} \tag{3.20}$$

where by $\mathrm{d} f(\phi_c)$ we denote the endomorphism of the cotangent space to M^{2n} at ϕ_c, $T^* M_{\phi_c}$, induced by f.

In fact we can think of f as a diffeomorphism of M^{2n} preserving the Kähler structure and taking L_0 to L_1. Near a fixed point $\mathrm{d} f(\phi_c)$ has the form

$$\mathrm{d} f(\phi_c) = e^{2\phi J A(\phi_c)}, \tag{3.21}$$

where $A(\phi_c)$ is given by (3.11). Acting on $\Lambda^{0,1} = \mathrm{span}\{\mathrm{d}\bar{z}, \ldots, \mathrm{d}\bar{z}_n\}$ and because $T^* M_{\phi_c} \otimes \mathbf{R} = T^* M_{\phi_c} \oplus \overline{T^* M}_{\phi_c}$, we have

$$\det_{\mathbf{R}}(I - \mathrm{d} f(\phi_c)) = \det_{\mathbf{C}}(I - \overline{\mathrm{d} f}(\phi_c)) \det_{\mathbf{C}}(I - \mathrm{d} f(\phi_c))$$

(2) Although $\mathcal{A}(\phi_c)$ is zero for a constant critical point we consider here, as related to each critical point, the area shown in Fig. 3(a).

with the important result that:

$$\det_{\mathbf{R}}(I - \mathrm{d}f(\phi_c)) = \prod_{i=1}^{n} \left(1 - e^{-i\pi\alpha_i(\phi_c)}\right)\left(1 - e^{i\pi\alpha_i(\phi_c)}\right) \tag{3.22}$$

Simili modo, it is easy to see that

$$Trf^*\Lambda^{0,\mathrm{even}} - Trf^*\Lambda^{0,\mathrm{odd}} = \prod_{i=1}^{n} \left(1 - e^{-i\pi\alpha_i(\phi_c)}\right)$$

by considering the bundle as a product of $U(1)$ bundles, and we obtain

$$L_{\mathrm{Dol}}(f) = \sum_{\text{fixed points}} \prod_{i=1}^{n} \frac{1 - e^{-i\pi\alpha_i(\phi_c)}}{\left(1 - e^{-i\pi\alpha_i(\phi_c)}\right)\left(1 - e^{i\pi\alpha_i(\phi_c)}\right)} \,. \tag{3.23}$$

Comparison with (3.18) tells us that the topological invariant provided by the partition function is therefore the following: $\mathbf{Z}(-\rho)$ is the Lefschetz number of the manifold.

A slightly different version of the model allows another interpretation for the partition function as a topological invariant. If one considers closed loops in M^{2n}, instead of imposing boundary conditions on Lagrangian submanifolds, the partition function is

$$\mathbf{Z}(k) = \int_{M^{2n}} \mathrm{d}\,\mathrm{vol} \prod_{i=1}^{n} \frac{\alpha_i(m)}{\left(1 - e^{i\pi\alpha_i(m)}\right)} \tag{3.24}$$

where the $\alpha_i(m)$ are now the first Chern classes of the line bundles in which the tangent bundle of M^{2n} decomposes. The set of critical points is now the manifolds itself; the integrand is the Todd class, and the partition function is the arithmetic genus. There are two differences with respect to formula (3.18): a) there is only n constant eigenfunctions of the operator $K = J\left(\frac{\mathrm{d}}{\mathrm{d}x} + T^+ \otimes T\right)$; in the previous case there were $2n$, and so we obtain the factor in the numerator. b) It is not natural now to consider a special m as reference point and $\mathcal{A}(m) = 0$, $\forall m$. In the case where $M^{2n} = S^2$, this situation, both in physical and mathematical aspects, has been deeply studied in reference [32].

A supersymmetric extension of the fixed point version of the model has been dealt with by Blau [33] to describe the symplectic invariants recently introduced by Weinstein [34] within the framework of Quantum Mechanics. The idea is that a rotation with $\alpha_i = 2\pi$, $\forall I$, gives the symplectic transformation $-I$. if $\mathrm{Sym}\, M^{2n}$ denotes the space of diffeomorphisms preserving the symplectic form ω, the partition function provides an invariant for elements in $H^1(\mathrm{Sym}\, M^{2n}, \mathbf{R}/\Gamma)$, where Γ is the period group of ω, because we are dealing with loops in $\mathrm{Sym}\, M^{2n}$, which are families of loops in M^{2n} itself, when taking L_0 to $L_1 = L_0$.

In the case of the manifold being strictly homogeneous of the form G/T where T is a maximal torus, e.g. $Sp(2n)/T^n$, the quantization of a similar model, based on the choice of Darboux variables, has been carried out in Reference [35] leading to a path integral version of the co-adjoint orbit method of building irreducible representations. The difference lies in the boundary conditions but the choice of isolated critical points is due to the addition of a hamiltonian vector field. Alvarez, Singer and Windey [36] also offer a Quantum Mechanical version of the Borel-Weyl theorem by supersymmetric path integral methods. Their model is related with the model considered in this paper: an extension of the WZW topological quantum mechanics by $2n$ Lagrange multipliers, which play the rôle of the A_0 potencial in Gauge Theories, exhibits gauge symmetry under $\mathrm{Maps}(I,\mathcal{H})$, where $\mathcal{H}$ is the Heisenberg group. $BRST$ fixing this symmetry in the extended model in topological quantum mechanics one arrives in the supersymmetric quantum mechanical model considered in Reference [36]. A detailed study of the ideas in the present work as compared with the three references quoted above will be the subject of a separate publication. It is worthwhile to mention that, when $M^{2n} = G/T$, (3.18) is the Weyl character formula, (5.15) in [36], if we do not consider the Jacobian of the $SO(2)$ action on the choice of normal coordinates.

The invariance under continuous deformations of the partition function allowing these interpretation as a topological invariant is by no means evident a priori. To study the dependence of $\mathbb{Z}(k)$ on the metric and the almost complex structure one must consider the functional integral,

$$\frac{\delta\mathbb{Z}(k)}{\delta J_{ij}(\psi)} = ik\int [\mathrm{d}\phi]''\delta(\phi-\psi)''(\dot{\phi},\phi')_{ij}e^{ikA(\phi)}\ , \tag{3.25}$$

which is almost impossible to deal with. In the Stationary Phase Approximations, however, it is zero.

$$\frac{\delta\mathbb{Z}(k)}{\delta J_{ij}(\psi)} \overset{k\to\infty}{\simeq} \mathrm{id}\sum_{\mathrm{Crit}\,A}(\dot{\phi},\phi')_{ij|_{\phi_c}}\cdot e^{ikA(\phi_c)}\cdot\frac{\mathrm{Pfaff}\ \Delta_{\mathrm{Ghosts}}}{\mathrm{Det}^{1/2}\Delta_B^G}\cdot e^{\frac{i\pi}{4}\eta_{h(\phi_c)}(0)} \tag{3.26}$$

because the critical points are constant loops and

$$(\dot{\phi},\phi')_{ij|_{\phi_c}} = \int_{-\infty}^{\infty}\mathrm{d}\tau\int_0^1\mathrm{d}x\left(\frac{\partial u^i}{\partial x}\frac{\partial u^j}{\partial\tau}-\frac{\partial u^j}{\partial x}\frac{\partial u^i}{\partial\tau}\right)\bigg|_{\phi_c} = 0\ ,$$

checking with the topological significance of $\mathbb{Z}(k)$ previously found for k large. An indirect argument can be given for this property holds to any order in the expansion in $\frac{1}{k}$. The integral in (3.27) is nothing but the expectation value of the torsion of the «curve» at the ground state, easy to see by choosing appropriate coordinates. It happens that, the Hamiltonian being zero, all the quantum states are ground states. Geometric quantization of the classical phase space M tells us that the Hilbert space is the space of sections of a line bundle on M restricted to a lagrangian submanifold. Essentially, the states are linear combinations of delta functions, centred at the intersecting points to give a unique quantization procedure. Therefore, it is natural to guess that the average of the quantum fluctuations of the torsion at such a kind of states is zero.

3.3. Knot invariants

The addition of a «source» term to the area functional modifies the variational equations to

$$J^i_j[\phi]\frac{du^j}{dx} = \eta^i(x) . \tag{3.27}$$

To analyze this more complicated problem it is convenient to use an approach based on the physical context of fractionary, braid, statistics [37]. Consider a system of n indistinguishable particles moving on a compact Riemann surface Σ. The configuration space is

$$M^{2n} = \frac{\Sigma_{(1)} \times \Sigma_{(2)} \times \ldots \times \Sigma_{(n)} - D}{S_n};$$

we subtract in the direct product the points where two or more particles coincides and identify points which are permutations in the symmetric group S_n of the n particles. The assumption is that our manifold M^{2n} is now the configuration space of n indistinguishable particles, each of them restricted to moving in Σ. The statistic, allowing only homologically trivial trajectories in Σ, are in the two-dimensional case, determined by the one-dimensional unitary representations of the braid group, if $\Sigma = S^2$, $\pi_1(M^{2n}) = B_n$, as opposed to higher dimensional cases where the pertinent group is S_n.

The action governing the dynamics is

$$\begin{aligned} S &= \int dx \left\{ J_{ij}[\phi]u^i \frac{du^j}{dx} + \eta_i(x)u^i(x) \right\} \\ &= \int dx \left\{ \sum_{\alpha=1}^{n} \left(J_{\alpha ab}[\phi]u^a_\alpha \frac{du^b_\alpha}{dx} + \eta_{\alpha a}(x)u^a_\alpha(x) \right) \right\} \end{aligned} \tag{3.28}$$

$$\alpha = 1,\ldots,n, \qquad a,b = 1,2 .$$

The second form of (3.28), coming from the Darboux theorem, explicitely shows how our dynamical system can be understood as a particle moving in $\mathbb{C}^n$, a local chart in M^{2n}, or n particles moving in $\mathbb{C}$, a local chart in Σ.

Special characteristics of the action in (3.28) are that M^{2n} is also the phase space and the whole dynamics comes from the source term.

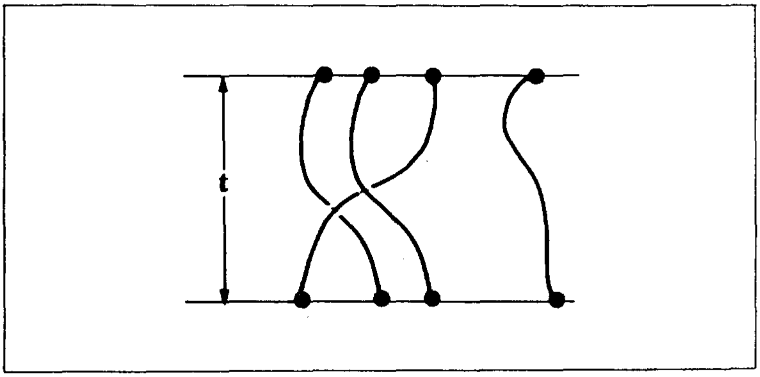

Fig. 4. A closed loop in the phase space of four indistinguishable particles representing a braid.

The particular choice of «point» sources of the kind $\eta_{\alpha a}(x) = \delta(x - x_\alpha) b_{\alpha a}$ where the x_α lie in the unit interval and are parametrized by a «evolution» parameter $t \in [0, 1]$ provides a term in the integrand of the partition function of the form

$$\beta_n(t) = e^{i \sum_{\alpha=1}^{n} b_{\alpha a} u^a_\alpha [x_\alpha(t)]} \; ; \tag{3.29}$$

this is a map from the interval $[0, 1]$ to the set of n unordered points in $\mathbb{C}$ which can be interpreted as a braid if $\beta_n(0) = \beta_n(1)$. The periodicity condition requires that $u^a_\alpha[x_\alpha(0)] = u^a_\rho[x_\rho(1)] = c^a$, Fig. 4, for some α and ρ, not-necessarily different, between 1 and n. The «quantization» condition follows

$$b_{\alpha a} = b_{\rho a} + 2\pi \frac{n_a}{c^a}, \qquad n_a \in \mathbb{Z} . \tag{3.30}$$

The corresponding quantum operator

$$\hat{\beta}_n(t) = e^{i : \sum_{\alpha=1}^{n} b_{\alpha a} \hat{u}^a_\alpha [x_\alpha(t)] :} \tag{3.31}$$

with a normal order prescription denoted by $::$, is a «blip-vertex» operator because of the commutation relations between the $\hat{u}'s$: $[\hat{u}^a_\alpha(x), \hat{u}^b_\alpha(x)] = J^{ab}_\alpha$. Because the phase space is compact, the Hilbert space is finite dimensional and all the states have zero energy due to the topological character of the action. Therefore, $Tr\hat{\beta}_n(t) = \sum_{\psi_0} \langle \psi_0 | \hat{\beta}_n | \psi_0 \rangle$ is the expectation value given by

$$Tr\hat{\beta}_n(t) = e^{i \sum_{\alpha=1}^{n} \sum_{\rho=1}^{n} G(x_\alpha(t), x_\rho(t))} \tag{3.32}$$

where

$$G(x_\alpha(t), x_\rho(t)) = \langle : \hat{u}^a[x_\alpha(t)] \hat{u}^b_\rho[x_\rho(t)] : \rangle b_{\alpha a} b_{\rho b}$$

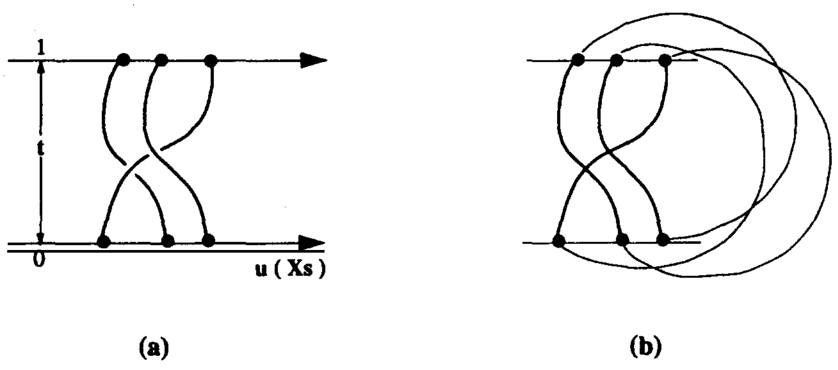

Fig. 5. The three basic elements of the braid group.

is the Green function describing the motion of a particle from $x_\alpha(t)$ to $x_\rho(t)$. This can be computed in terms of the Path Integral

$$\langle : \hat{u}^a_\alpha[x_\alpha(t)]\hat{u}^b_\rho[x_\rho(t)] : \rangle = \frac{1}{\mathbb{Z}(k)} \times \int [\mathrm{d}\,u^a_1]\ldots[\mathrm{d}\,u^a_n]u^a_n[x_\alpha(t)]u^b_\rho[x_\rho(t)]e^{ik\int \mathrm{d}x \sum_{\alpha=1}^{n} J^{ab}_\alpha u_{\alpha a}\frac{\mathrm{d}u_{\alpha b}}{\mathrm{d}x}} \tag{3.33}$$

by means of the Stationary Phase Approximation, to find

$$Tr\hat{\beta}_n(t) = e^{\frac{1}{2}\sum_\alpha\sum_\rho\left[\{b_{\alpha a}b_{\rho b}-b_{\alpha b}b_{\rho a}\}\right]J^{ab}_\alpha\varepsilon(x_\alpha(t)-x_\rho(t))} \tag{3.34}$$

where $\varepsilon(x)$ denotes the sign function.

Because a link is the closure of a braid, Fig. 5, one hopes that the expectation value of the operator $\hat{\beta}_n$, which arises from the quantization of β_n , will be a topological invariant tied to links in M . In fact, we have that: $Tr\hat{\beta}_n(t)$ gives the quantum evolution in the Heisenberg representation as a holonomy with values in a discrete $\mathbb{Z}_2$ group described by (3.34). For $t = T_f$ the final state is reached and the invariant associated to the braid is

$$Tr\hat{\beta}_n(T_f)/Tr\hat{\beta}_n(T_i) = e^{i\sum_\alpha\sum_\rho\{G(x_\alpha(T_f),x_\rho(T_f))-G(x_\alpha(T_i),x_\rho(T_i))\}} . \tag{3.35}$$

For $n = 2$ the three basic cases are drawn in Fig. 7: 1) if $x_\alpha(T_i) > x_\rho(T_i)$ and there are no crosses we obtain the idently element. 2) if $x_\alpha(T_i) > x_\rho(T_i)$, there is one cross and $x_\alpha(T_f) < x_\rho(T_f)$, (3.35) gives the invariant of the generator of the braid group, σ . 3) if $x_\alpha(T_i) < x_\rho(T_i)$, there is one cross and $x_\alpha(T_f) > x_\rho(T_f)$ we get the invariant of the inverse, σ^{-1} .

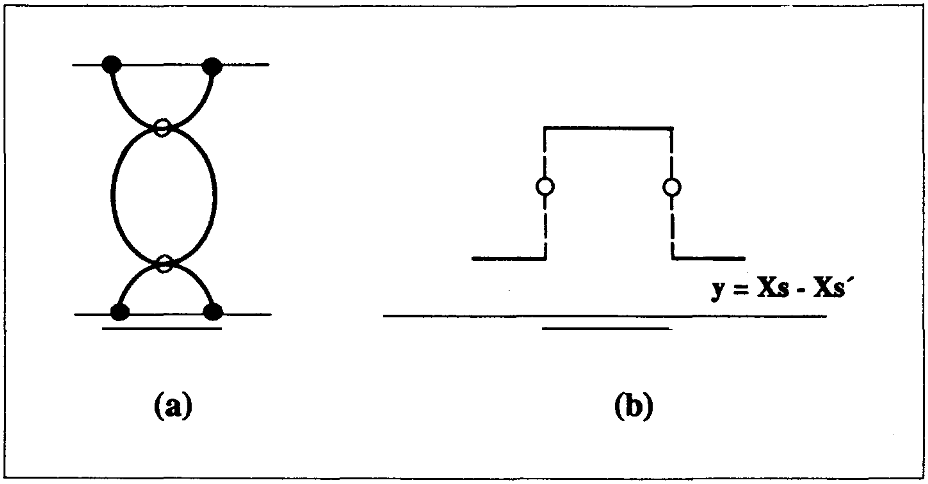

Fig. 6. A braid in N and its closure as a knot.

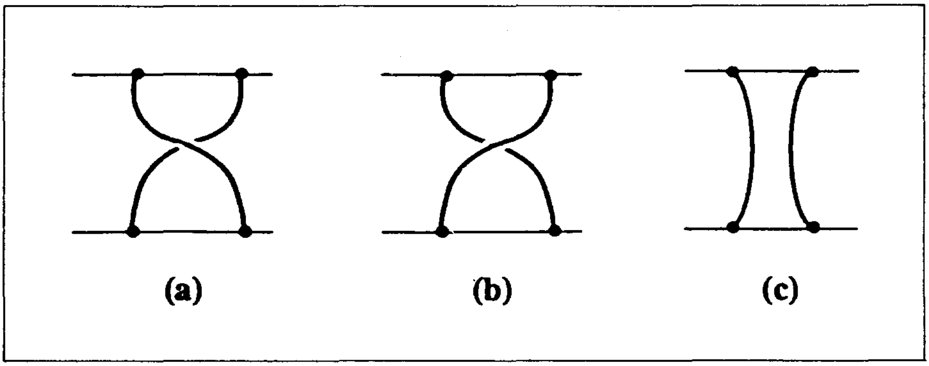

Fig. 7. Jumps in the correlation function appearing at intersectiong points of the braid.

Physically the «magnetic» susceptibility χ_{ij} of a material, a zero dimensional magnetic system, when it is pierced by point sources of magnetic fields, appears in the exponent as a $\mathbb{Z}_2$ connection,

$$\chi_{ij} = \sum_{\alpha=1}^{n} \sum_{\rho=1}^{r} J_{ij}[\phi_c]\varepsilon(x_\alpha - x_\rho) . \tag{3.36}$$

This suggests a one-dimensional vantage point of the Polyakov linking number [33]. In the plane projection of a link there are under and over crossings; the magnetic susceptibility counts the number of crossings with signs, giving the linking number in terms of the order of the points where the crossings are projected in the unit interval, Fig. 8.

In the previous discussion we have not considered the problem arising when $x_\alpha =$

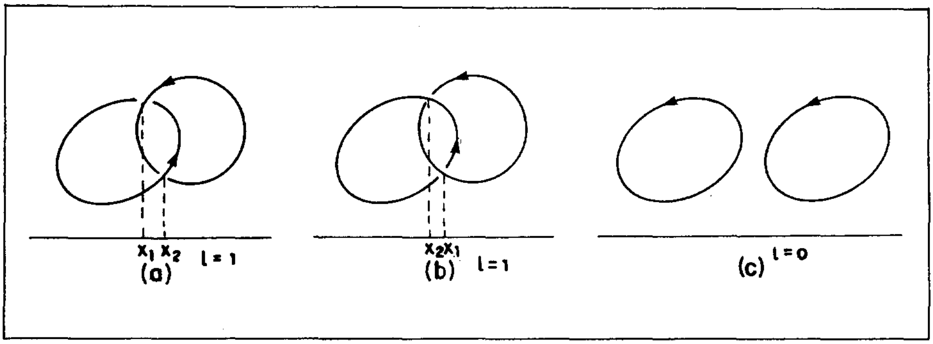

Fig. 8. Linked loops in N and its projection on the unit interval.

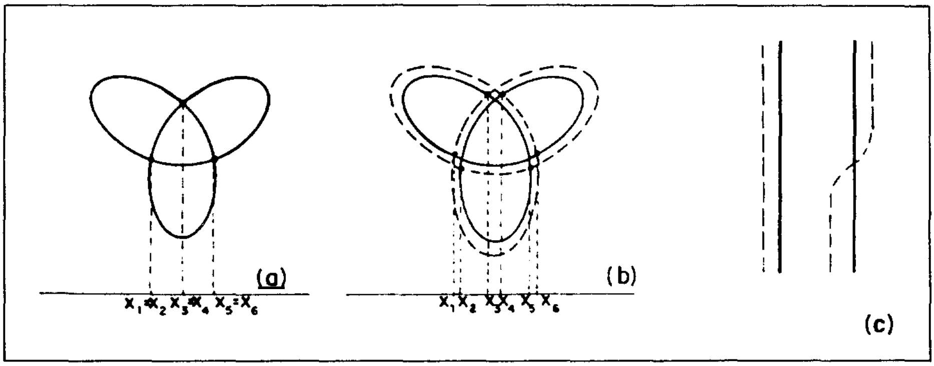

Fig. 9. Framing the trefoil knot.

x_ρ. The obvious loophole is to regularize the sign function near zero by

$$\varepsilon(x_\alpha - x_\rho) = \lim_{\delta \to 0} tanh \frac{x_\alpha - x_\rho}{\delta} ;$$

in such a way one obtains a well defined magnetic susceptibility even in the points where two sources coincides but yielding the same holonomy as in (3.39). Another possibility, appropriate for dealing with self-linking situations, is to consider a «framing», Fig. 9, of the link in M. The ambiguity arising when framing with different twisting numbers are considered is tamed by knowing that new factors of the (3.29) kind appear at the new crossings. Thus we can also get invariants of knots: self-linking properties can be seen as linking numbers of two knots related by a framing.

The previous discussion is close in spirit to the description of knot and link invariants in a Abelian Chern-Simons theory with sources as explained in Reference [39]. Another approach is interesting. In [40] Wilson lines are represented as path integrals over the loop space of a co-adjoint orbit G/T, e.g. $Sp(2n)/T^n$, with an action which is the Wess Zumino-Witten term. We could consider our manifold M^{2n} as the direct product of the moduli space of flat G bundles over a Riemann surface with a co-adjoint orbit G/T corresponding to a given irreducible representation; we envision only a Wilson loop. In the Chern-Simons gauge theory we attempt to compute

(3.36) $$\mathbf{Z}(M^3;C) = \int_{\mathcal{A}/\mathcal{G}} \mathcal{D}A\, TrP(e^{i\oint_C A})e^{i\frac{k}{8\pi}\int_{M^3}(A\mathrm{d}A+\frac{2}{3}A^3)}$$

when $M^3 = \sum \times I$, in the temporal gauge $A_0 = 0$; the solution of the constraint equation

(3.37) $$*F_A^{[2]} = \delta(x - x_0)\ ,$$

replacing the Lie algebra generators in $F_A^{[2]}$ by classical variables in the co-adjoint orbit, $Sp(2n)/T^n$, leads to an action of the form:

(3.38) $$S = \int \mathrm{d}\,x J_{mn}[\phi]\alpha^m \frac{\mathrm{d}\,\alpha^n}{\mathrm{d}\,x} + \int \mathrm{d}\,x J_{ij} u^i \frac{\mathrm{d}\,u^j}{\mathrm{d}\,x} + \text{ constant}$$

where $m, n = 1, 2, \ldots, (2g-2)$ dim G, $i, j = 1, \ldots,$ dim G/T and g is the genus of Σ. The first term in (3.38) is the WZW functional over the moduli space of flat connections, and is there even in the absence of flux lines; the second term is the same over the co-adjoint orbit of G/T; the third, constant, comes from the boundary conditions of C and a possible cobordism implied by M^3 in the event of it not being a cylinder but rather a manifold with boundary $\partial M^3 = \Sigma \cup \Sigma'$. The full quantum theory is recovered by the path integral

(3.39) $$\mathbf{Z}(k) = \int_{\mathcal{M}\times G/T} [\mathrm{d}\,\alpha^m][\mathrm{d}\,u^i]e^{ikS}$$

providing a bridge between Chern-Simons gauge theory and topological quantum mechanics. Briefly, it assumes the quantization of Chern-Simons theory via functional integral methods in the Hamiltonian formalism, after imposing constraints.

In the case of Σ being equal to S^2, the moduli space is a point, and we are left with the Borel-Weyl-Bott theory of the irreducible representations of a Lie group as described by path integrals over co-adjoint orbits. We could also add non-abelian sources; we have a distinct η for each different representation, and repeat the discussion of the, in this case coloured, braid statistics. This richer interpretation of the invariants in (3.35) for now coloured braids contact with the Witten-Jones invariants of Chern-Simons theory.

4. STOCHASTIC QUANTIZATION AND PHYSICAL APPLICATIONS

In this Section our aim is to show how the models in Section §2 and §3 are related and also to consider physical applications of the ideas previously developed to condensed matter physics and the topological understanding of spin.

4.1. Stochastic quantization and $BRST$ symmetry

The first task is to discuss the stochastic quantization of the Wess-Sumino-Witten topological quantum mechanics, performed, as explained in Reference [34], by the supersymmetric sigma model described in §2 . We shall apply the conventional supersymmetric presentation of stochastic quantization to our model at the same time going a further step in considering the problems which appear when, as in our case, there is supersymmetry breaking. The following developments explain, together with §3.1, the schizöid behaviour of the area (Wess-Zumino-Witten) functional: in Section §2 the WZW functional plays the rôle of the superpotential in a Supersymmetric Quantum Theory, the standard recipe for introducing interactions without spoiling supersymmetry. In Section §3 the WZW funcional is the action functional for a model in Quantum Mechanics. The link between the two disguises of the WZW functional is called Stochastic Quantization.

Recall that the action for WZW topological quantum mechanics is the area funcional

$$\mathcal{A}[\phi,x,\tau)] = \langle\phi'(x,\tau), J[\phi]\phi(x,\tau)\rangle .$$

Although it is independent of the τ-parameter, we shall consider τ as a stochastic time induced by a white noise, a Gaussian random variable, $b(x,\tau)$. From the solutions of the Langevin equation

$$\frac{\partial\phi_b}{\partial\tau} + sJ[\phi_b]\frac{\partial\phi_b}{\partial x} = b(x,\tau) \tag{4.1}$$

the stochastic partition function is defined

$$\mathbf{Z}_{\text{Stoch}} = \int [\mathrm{d}\,b][\mathrm{d}\,\phi]''\delta(\phi-\phi_b)''\exp\left\{-\frac{1}{2}\langle b(x,\tau), b(x,\tau)\rangle\right\} . \tag{4.2}$$

To perform the functional integral in the stochastic variable b it is possible to use the Langevin equation to change the argument of the Dirac delta functional

$$\begin{aligned} ''\delta(\phi-\phi_b)'' &='' \delta\left(\frac{\partial\phi}{\partial\tau} + sJ[\phi]\frac{\partial\phi}{\partial x} - b(x,\tau)\right)'' \det\, M \\ \det\, M &= \det\left[\left(\frac{\partial}{\partial\tau} + sJ[\phi]\frac{\partial}{\partial x}\right)\right]\delta(\tau'-\tau) , \end{aligned} \tag{4.3}$$

in a convenient way. The result is

$$\begin{aligned} \mathbf{Z}_{\text{Stoch}} &= \int [\mathrm{d}\,\phi] \det\, exp\left\{-\frac{1}{2}\langle\phi',\phi'\rangle_\beta + \frac{s^2}{2}\langle\dot\phi,\dot\phi\rangle_\beta\right\} \\ &\cdot \exp\left\{-s\left[\mathcal{A}(\beta) - \mathcal{A}(0)\right]\right\} \\ \langle f_1, f_2\rangle_\beta &= \int_0^\beta \mathrm{d}\,\tau \int_0^1 \mathrm{d}\,x \left(g^{ij}(x,\tau) f^*_{1i}(x,\tau) f_{2j}(x,\tau)\right) \end{aligned} \tag{4.4}$$

and writing the determinant as a functional fermionic integral we finally get

$$\mathbf{Z}_{\text{Stoch}} = \int [d\phi][d\xi][d\eta] e^{-\int_0^\beta d\tau I_{\text{Stoch}}} e^{-sI_{\text{Top}}} ; \tag{4.5}$$

here I_{Stoch} and I_{Top} are

$$\begin{aligned} I_{\text{Stoch}} &= \int_0^1 dx \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}, \frac{\partial\phi}{\partial\tau}\right) + \frac{s^2}{2}\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial x}\right) \right. \\ &\quad \left. + \left(\xi, \frac{\partial\eta}{\partial\tau} + sJ[\phi]\frac{\partial\eta}{\partial x}\right)\right] \\ I_{\text{Top}} &= \int_0^\beta d\tau \int_0^1 \left(\frac{\partial\phi}{\partial\tau}, J[\phi]\frac{\partial\phi}{\partial x}\right); \quad (a,b) = g_{ij}a^i b^j \end{aligned} \tag{4.6}$$

and ξ, η are the Grassman variables of Section §2. Note that I_{Stoch} is the classical action of the supersymmetric sigma model of the Section §2 with supersymmetry generators

$$\begin{aligned} Q_s &= \int_0^1 dx \left(\xi, \frac{\partial\phi}{\partial\tau} + sJ[\phi]\frac{\partial\phi}{\partial x}\right); \\ Q_s^* &= \int_0^1 dx \left(\eta, \frac{\partial\phi}{\partial\tau} + sJ[\phi]\frac{\partial\phi}{\partial x}\right) . \end{aligned} \tag{4.7}$$

In a stochastic process the topological action arises in a natural manner. After integration of the random variable $b(x,\tau)$ a dependence of the boundary values, $\phi(x,0)$, $\phi(x,\beta)$, must appear. Moreover I_{Top} is the essential term, because I_{Stoch} comes from the applications of the $BRST$ procedure to fixing a gauge in I_{Top}. The gauge freedom of I_{Top}, $\phi(x,\tau) \to \phi(x,\tau) + \varepsilon(x,\tau)$ for arbitrary ε such that $\varepsilon(x,0) = \varepsilon(x,\beta) = 0$, essential to its topological character, requires use of the $BRST$ quantization procedure to deal with a model with dynamics governed by I_{Top}. In terms of the $BRST$ operator, the fermionic operator a defined by

$$\begin{array}{lll} a\phi(x,\tau) = \varepsilon(x,\tau) & a\eta(x,\tau) = \lambda(x,\tau) & a^2 = 0 \\ ; & ; & , \\ a\xi(x,\tau) = 0 & a\lambda(x,\tau) = 0 & [\frac{\partial}{\partial\tau}, a] = 0 \end{array} \tag{4.8}$$

the gauge fixing action is

$$I_{GF} = \int_0^1 dx\, a\left(\xi(x,\tau), \partial\phi\partial\tau + sJ[\phi]\partial\phi\partial x + \frac{\lambda}{2}(x,\tau)\right)$$

or using (4.8),

$$I_{GF} = \int_0^1 \mathrm{d}x \left[\frac{1}{2}(\lambda,\lambda) + \right. \tag{4.9}$$
$$\left. + \left(\lambda, \frac{\partial\phi}{\partial\tau} + sJ[\phi]\frac{\partial\phi}{\partial x}\right) - \left(\xi, \frac{\partial\eta}{\partial\tau} + sJ[\phi]\frac{\partial\eta}{\partial x}\right]\right) .$$

The standard $BRST$ quantization procedure is given by a functional integral of the action $I[\phi,\xi,\eta,\lambda] = I_{\text{Top}} - I_{GF}$ which, integrating in the Lagrangian multiplier field $\lambda(x,\tau)$, is equivalent to I_{Stoch}

$$\int [\mathrm{d}\phi][\mathrm{d}\xi][\mathrm{d}\eta][\mathrm{d}\lambda] e^{-\int_0^\beta \mathrm{d}\tau I[\phi,\xi,\eta,\lambda]} = \tag{4.10}$$
$$= \int [\mathrm{d}\phi][\mathrm{d}\xi][\mathrm{d}\eta] e^{-\int_0^\beta \mathrm{d}\tau I_{\text{Stoch}}[\phi,\xi,\eta]}$$

In particular the gauge condition $\lambda(x,\tau) = \frac{\partial\phi}{\partial\tau} + sJ[\phi]\frac{\partial\phi}{\partial x}$ means that the $BRST$ symmetry of the quantization procedure is precisely the supersymmetry (4.7)

$$\mathcal{A}(x,\tau) = \left(\xi(x,\tau), \frac{\partial\phi}{\partial\tau} + sJ[\phi]\frac{\partial\phi}{\partial x}\right) . \tag{4.11}$$

If we perform first the fermionic functional integral, à la Nicolai [34], we obtain,

$$\mathbf{Z}_{\text{Stoch}} = \int [\mathrm{d}\phi][\mathrm{d}\lambda] \det \left\{\frac{\partial}{\partial\tau} + sJ[\phi]\frac{\partial}{\partial x}\right\} \exp - \int_0^\beta \mathrm{d}\tau$$
$$\int_0^1 \mathrm{d}x \left[\frac{1}{2},(\lambda,\lambda) + \left(\lambda, \frac{\partial\phi}{\partial\tau} + sJ[\phi]\frac{\partial\phi}{\partial x}\right]\right)$$

and realizing that the determinant is the Jacobian of the map

$$\Phi(x,\tau) = \frac{\partial\phi}{\partial\tau} + sJ[\phi]\frac{\partial\phi}{\partial x}, \tag{4.12}$$

finally arrive at the apparently simple expression

$$\mathbf{Z}_{\text{Stoch}} = \int [\mathrm{d}\Phi] e^{-\int_0^\beta \mathrm{d}\tau \int_0^1 \mathrm{d}x \{\frac{1}{2}(\Phi,\Phi)\}} \tag{4.13}$$

after integrating in λ. $\mathbf{Z}_{\text{Stoch}}$ is thus a functional integral of the Gaussian character but highly non-trivial due to the non-local character of the map (4.12). Furthermore, if the dimension of the kernel of the operator $\frac{\partial}{\partial\tau} + sJ[\phi]\frac{\partial}{\partial x}$ is different from zero the map (4.12) is not one-to-one and the final expression (4.13) is non sense. This situation happens precisely when the dimension of the moduli space of the instantons is different from zero, exactly our own case when M is non-trivial. The failure of the Nicolai map, related to supersymmetry breaking, is also related to a ill-defined stochastic process.

In stochastic quantum mechanics one considers a stochastic flow by introducing a Hamiltonian ruling the τ-evolution. In our case we have

$$H = \int_0^1 dx \left[\frac{1}{2}(\pi_\phi, \pi_\phi) + \frac{s^2}{2}\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial x}\right) - s(\xi, J[\phi]\eta)\right] \tag{4.14}$$

which, upon quantization,

$$\begin{gathered} \pi_\phi = i\frac{\delta}{\delta\phi} \ ; \qquad \xi = \psi \ ; \qquad \eta = \chi \\ \left[\phi(x,\tau), \pi_\phi(x',\tau)\right] = i\delta(x - x') \ ; \\ \{\psi(x,\tau), \eta(x',\tau)\} = \delta(x - x') \end{gathered} \tag{4.15}$$

becomes the Hamiltonian of the supersymmetric sigma model. The supersymmetry appears as carried out by the $BRST$ quantum operators

$$\begin{aligned} \hat{Q} &= \int_0^1 dx \left(\psi, i\frac{\delta}{\delta\phi} + sJ[\phi]\frac{\partial\phi}{\partial x}\right) ; \\ \hat{Q}^+ &= \int_0^1 \left(\chi, i\frac{\delta}{\delta\phi} + sJ[\phi]\frac{\partial\phi}{\partial x}\right) \end{aligned}$$

such that $\hat{Q}^2 = \hat{Q}^{+2} = 0$ and $H = \hat{Q}^+\hat{Q} + \hat{Q}\hat{Q}^+$. The symplectic from $\omega = (\delta\eta, \delta\xi) + (\delta\phi, \delta\pi_\phi)$ induces a Hamiltonian flow $i_{\frac{\partial}{\partial\tau}}(\omega) = \delta H$ and, according to the discussion of the Section §2, the existence of instantons in the flow produces supersymmetry breaking and/or Floer homology groups which are non-trivial.

Supersymmetry breaking in stochastic quantization is better described by the Fokker-Planck equation,

$$\frac{\partial}{\partial\tau}\bar{P}[\phi,\tau] = \mathcal{H}\bar{P}[\delta,\tau] \ , \tag{4.16}$$

where $\bar{P}[\phi,\tau]$ is the conditional probability of $\phi(x,\tau)$ having a value ϕ at time τ if at 0 the value was zero, and $\mathcal{H}$ is the Hamiltonian density of (4.14). Writing $\bar{P}$ in the form $\bar{P}[\phi,\tau] = P[\phi,\tau]\exp\{-s\mathcal{A}(\phi)\}$, it is clear that for large τ $\exp\{-s\mathcal{A}(\phi)\}$ dominates P because it is a zero energy eigenfuctional of $\mathcal{H}$. In this way, the exponential of the WZW functional appear naturally in the large time behaviour of the supersymmetric sigma model for which the WZW functional was the superpotential! Àlas, when the manifold M presents more than one critical point, our old «global ground state», $e^{-sA\phi}$, of Section §3 is not a good integration measure; it is in fact non-normalizable, and the stochastic process is ill-defined.

To unveil more precisely the topological meaning of the non-uniqueness of the large τ behaviour of our stochastic system we must study the transition amplitude between different ground states, rather than the partition function which is non-well suited in non-ergodic processes. We have seen in Section §2 that in the stationary phase approximation it is given by

$$\langle \phi_{c_b} | e^{-\beta H} | \phi_{c_a} \rangle = \sum_{I \in \hat{M}(\phi_{c_b}, \phi_{c_a})} \gamma(I) \frac{\text{Pfaff}\, \Delta_F}{\text{Det}^{1/2} \Delta_B} e^{-s|A(\phi_{c_b}) - A(\phi_{c_a})|} .$$

In normal coordinates along the geodesics «foliating» the holomorphic map ϕ_I, fixing to zero the rotation freedom along the geodesic and factorizing the metric tensor, we are left with the operators

$$(4.17)\qquad \begin{aligned} \left[\Delta_F(\phi_I)\right]^i_j &= \left(\sigma^3 \otimes \delta^i_j \frac{\partial}{\partial \tau} + 1_2 \otimes J^i_j \frac{\partial}{\partial x}\right) \\ \left[\Delta_B(\phi_I)\right]^i_j &= \left(-\delta^i_k \frac{\partial}{\partial \tau} + J^i_k \frac{\partial}{\partial x}\right)\left(\delta^k_j \frac{\partial}{\partial \tau} + J^k_j \frac{\partial}{\partial x}\right) . \end{aligned}$$

In the canonical basis previously used for J the quotient of the determinants is

$$(4.18)\qquad K = \left\{ \prod_{i=1}^{2n} \prod_{k=0}^{\infty} \left\{ \frac{\text{Det}\left(\sigma^3 \frac{\partial}{\partial \tau} + \lambda_k^{(i)^2}\right)}{\text{Det}\left[\left(-\frac{\partial}{\partial \tau} + \lambda_k^{(i)^2}\right)\left(\frac{\partial}{\partial \tau} + \lambda_k^{(i)^2}\right)\right]} \right\} \right\}^{1/2}$$

where $\lambda_k^{(i)}$ are the eigenvalues of $J\frac{d}{dx}$, because the operators (4.17) are sums of τ-dependent and x-dependent terms. The factors in (4.18) have been computed for large τ in SUSY quantum mechanics [22] and the solution is

$$K \overset{\tau \to \infty}{\simeq} \text{Det}^{1/2}\left(J\left[\phi_{c_b}\right] \frac{d}{dx}\right) .$$

Taking into account the rotation freedom of the normal system of coordinates along the geodesics parametrizd by τ, as we did in Section §3, intead of (4.19), we obtain

$$(4.20)\qquad K \overset{\tau \to \infty}{\simeq} \frac{\text{Det}^{1/2}\left|J(\phi_{c_b})\left\{\frac{d}{dx} + (T^+ \otimes T)(\phi_{c_b})\right\}\right|}{\text{Det}^{1/2} J(\phi_{c_b}) T^+ \otimes T(\phi_{c_b})} \cdot e^{-i\frac{\pi}{2}\mu(\phi_{c_b})} .$$

The large τ, stochastic, limit of the transition amplitude between two critical points is therefore inverse of the topological term contributed by the highest critical point to the partition function (3.16) in WZW topological quantum mechanics. It is remarkable that the large τ behaviour of the coboundary operator giving the instanton homology of the symplectic manifold is directly related with the Lefschetz formula deduced in topological quantum mechanics.

Two final comments about the connection between supersymmetric and topological sigma models: 1) the signs appearing in the definition of the instanton coboundary operator in the first, $(1+1)$-dimensional, model have a countepart in the topological, $(0+1)$-dimensional model. Computing the partition function in the topological model by the steepest descent method we must sum over all the steepest descent paths through each critical point; the contribution of every path will include a factor ± 1 if the orientation of the tangent space along the path is the same as the orientation of he tangent space to M at ϕ_c or not. 2) Since the contribution of the instantons is essentially captured at large k, one can guess that the topological expression of $\mathbb{Z}$ is true, no more corrections, even for non-large k.

4.2. Supersymmetric sigma model and condensed matter

Coming now to possible physical applications of the models under discussion, a first possibility is to use for the description of systems in condensed matter physics in $(1+1)$-dimensions with order parameters taking values in M, the space of states of the system being a Kähler manifold; the spatial boundary conditions require that the order parameter at the ends of the space lie in Lagrangian submanifolds of M. With these premises they can afford a model for one-dimensional liquid crystals in the high-chirality limit [12], where the «gradient» energy dominates over the «bulk» energy. Alternatively, they provide a theoretical laboratory for testing peculiar properties of some rare ferromagnetic materials, although the low-dimensionality makes the model unrealistic.

The crucial property which allows one to make qualitative phisical predictions is the ground state structure. The Hamiltonian of our supersymmetric sigma model exhibits discrete symmetry, translations in the «lattice» of the critical points of M. The vacuum degeneracy corresponding to this symmetry is not fully removed by instantons and the broken symmetry is only partially restored, those tied to the critical points contributing to the Floer homology remaining as ground states. Many physical properties are given by the ground state structure and these are well suited to describe blue phases of crystalline liquids in the high-chirality limit (see [12]). For instance, the ground state structure in the case of M being a Riemann surface of genus g is given by the Floer polynomia $F_t(\Sigma_g) = \Sigma_{\lambda_p} t^{\lambda_p} \dim HF^p(\Sigma_g) = 1 + 2gt + t^2$, Fig. 10.

Another physically significant matter is the fact that the supersymmetric sigma model we are dealing with is anomalous in the sense that the phenomenon of fermion fractionization takes place (see [13]). The choice of a critical point contributing to the Floer homology means that the system is in an equilibrium state which is a gaussian around the critical point *and* a Dirac sea, the fermionic state where all the negative energy eigen-

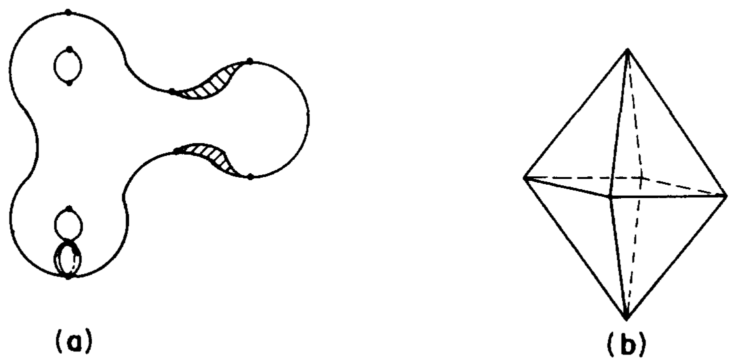

Fig. 10. Instantons and ground state structure for the sigma model with the Riemann surface of the Figure as the target manifold.

states are filled. The fermionic energy operator is the Hessian of the WZW action functional; it is evident that Dirac seas at critical points with relative Morse indices other than zero are different and, because we need to pick a ground state for quantizing the quantization procedure introduces an ambiguity in the definition of the fermionic number. To make this more precise, consider the operator

$$G = D_T e^{i\alpha \int dx \{ g_{ij}(\phi_c)\chi^i(x)\psi^j(x) \}} ; \tag{4.21}$$

D_T is the unitary operator representing in the Hilbert space the translation group of the «lattice» of critical points: $T = \mathbf{Z}_{2n+1} \otimes_{k=0}^{2n} \mathbf{Z}_{f_k}$, $f_k = \dim HF^k(M)$. The other factor in (4.21) is the unitary operator which adds a fermion number α to the fermionic content of any particular state. Starting from the ground state around the reference critical point $|\phi_0>_B$, no fermions at all, the vacuum orbit is $G|\phi_0>_B$. The degenerancy of the ground state, coming from the G action, yields a ill-definition of the fermion number:

$$\begin{aligned} &: \int dx \left\{ g_{ij}(\phi_c)\chi^i(x)\psi^j(x) \right\} : \{ |\phi_0>_B - g|\phi_0>_B \} = \mu_R(\phi_c) \\ &|\phi_c>_{F^{\mu_R}} = g|\phi_0>_{B=F^0} , \quad g \in G \end{aligned} \tag{4.22}$$

because the normal ordering is referred to $|\phi_0>_B$. Putting the same idea in another disguise, the concept of spectral flow re-appears. In the expansion of the fermionic fields in terms of eigenfunctions of the Hessian operator

$$\begin{aligned} \psi^i(x,\tau) = \sum_{\lambda_n} \big[& C_n^+(\phi_c) f_n^{+i}(x) \exp(-i\lambda_n^+\tau) + \\ & + C_n^-(\phi) f_n^{-i}(x) \exp(i\lambda_n^-\tau) \big] \end{aligned} \tag{4.23}$$

some coefficients C_n^+ corresponding to the positive part of the spectrum become C_n^-, in the negative part, when moving from one critical point to another. The number of those is precisely the relative Morse index.

The intelligence of the supersymmetric sigma model of Section §2 as the stochastic quantization of the topological sigma model of Section §3 suggests other physical applications. The norm of the «global» ground state of the first model is the partition function of the second one. In stochastic quantization we would say that the conditional probability relaxes at the equilibrium measure. Closing our eyes for a moment to the lack of ergodicity, inherent ot the existence of several ground states, physical properties at equilibrium in the first system are obtained as the correlation function of the second. For instance, the magnetic susceptibility given in (3.34)-(3.35) is the corresponding quantity in the supersymmetric sigma model when equilibrium is reached. The most important feature is that, having a dipolar structure, the magnetic susceptibility is independent of distance. The jump when the relative location of the «poles» is reversed reflects the topological obstructions in the problem; due to the topology of M there is no unique gaussian state around a unique critical point, the system is non-ergodic, and correspondingly one must take a local choice of the almost complex structure around each critical point, the jumps appearing at the intersection of the local charts. Staying close enough to a ground state the form of the magnetic susceptibility corresponds to dipole dissociation or Debye screening [14]; the model describes a plasma phase with a mass gap, at least for not too small temperatures (not too large β). In the analogous $(3+1)$-dimensional Yang-Mills system the Chern-Simmons partition function would describe a disorder electric phase at finite temperature, the Polyakov linking number giving a «perimeter law» [14] for the expectation value of the Wilson loops.

4.3. Topological theory of spin

A second possibility from the physical point of view is to use the WZW topological quantum mechanics as a topological theory of the spin of a relativistic particle moving in $D = 2n+1$ dimensional curved space-time manifold, along similar lines to those developed by Polyakov in [15]. Let X^μ be the local coordinates describing the motion (in euclidean time) of a particle in a universe X which locally is $\mathbf{R}^{2n+1}$. Consider the quantum action for a relativistic particle, including a functional integral in the velocities, given by

$$\mathbf{Z} = \int [\mathrm{d}X^\mu][\mathrm{d}h][\mathrm{d}\lambda][\mathrm{d}\phi]\, e^{-m\int_0^1 \mathrm{d}x h(x)}\, e^{i\int_0^1 \mathrm{d}x \lambda_\mu(\dot X^\mu - h e^\mu)}\, e^{ikA[\phi]}\,; \tag{4.24}$$

here λ is a Lagrange multiplier which upon integration enforces the constraint

$$e^\mu = h^{-1}(x)\dot X^\mu \tag{4.25}$$

where $h(x) = \left[g_{\mu\nu}(x)\dot{X}^{\mu}\dot{X}^{\nu}\right]^{1/2}$ is the «einbein» and $g_{\mu\nu}$ is the metric in X. From (4.25) it automatically follows that the «velocities» e^{μ} satisfy

(4.26) $$g_{\mu\nu}(X)e^{\mu}(X)e^{\nu}(X) = 1$$

and we will choose $g_{\mu\nu}(X)$ in such a way that (4.26) describes ouf Kähler manifold M. The spin factor introudced in [15] $\Phi(P)$, depending on a path P in X, is the partition function of WZW topological quantum mechanics

(4.27) $$\Phi(P) = \int [D\phi]e^{ikA(\phi)}$$

because when the particle moves along a closed loop P in X its velocity describes some path in M. For paths of the kind depicted in Fig. 11 our maps ϕ re-appear and the obvious topological meaning of $\phi(P)$ arises: self-intersections of the world-line of a particle with spin are fixed points of diffeomorphisms of the Kähler manifold where the «classical» spin takes values. The spin factor is essentially the expectation value of the «parallel transport» of the volume element of M along P. We compute this quantity in Section §3 by applying to the functional integral in (4.27) the stationary phase approximation method; the contribution at the fixed points is given by the ferminoc determinat

(4.28) $$\det\left\{\delta^i_j\frac{\mathrm{d}}{\mathrm{d}x}A^i_j(x;\phi_c)\right\} = \int[\mathrm{d}\xi][\mathrm{d}\eta]e^{\int_0^1 \mathrm{d}x\left\{\xi_i\left(\delta^i_j\frac{\mathrm{d}}{\mathrm{d}x}+A^i_j(x;\phi_c)\right)\eta^j\right\}},$$

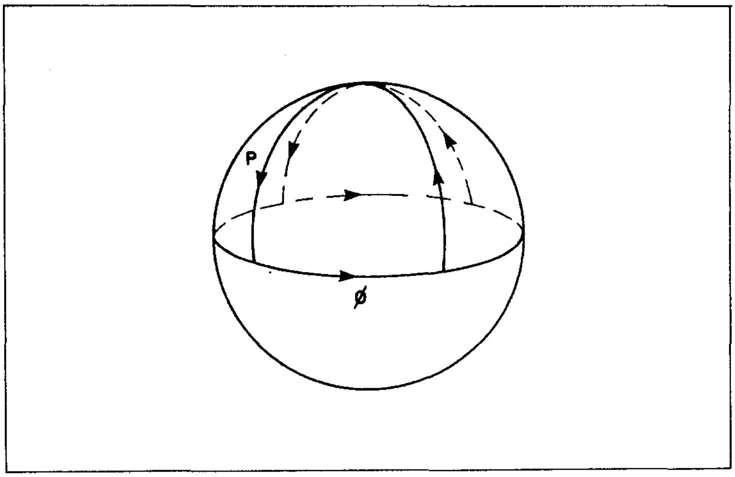

Fig. 11. A path described by the velocity vector in M when the particle moves in X.

leaving apart one phase and the «bosonic» determinant. Because ξ and η can be realized in the Hamiltonian formalism as γ-matrices it suggests that $\mathbb{Z}$ describes Dirac-type propagators more than Klein-Gordon ones.

To see that, explicitely, it is convenient to make a Fourier transform introducing constant sources $b^i(x) = p^i$ and computing

$$\left\langle e^{i\int_0^1 \mathrm{d}x(p,\phi(x))}\right\rangle = \int [\mathrm{d}\,\phi] e^{\frac{1}{\alpha}\int_0^1 \mathrm{d}x\left(\frac{\mathrm{d}\phi}{\mathrm{d}x},\frac{\mathrm{d}\phi}{\mathrm{d}x}\right)} e^{ikA[\phi]} e^{i\int_0^1 \mathrm{d}x(p,\phi)} \tag{4.29}$$

here we have included the «kinetic» term of the sigma model which introduces correlations between the velocities u^i. It is easier do the computation of (4.29) in the infrared limit; it requires a re-scaling of x to $y = Lx$. The dimensional constant $k_0 = k/L$ must then be quantized to be $k_0 = n/2L$ to give a well-defined functional integral. In the canonical system of coordinates tantamount to the splitting of the bundle (4.29) describes the quantum motion of a ϕ-particle moving on the manifold M in the presence of n Dirac monopoles sitting at the centres of n non-interesting fundamental two-cycles. In energy representation (4.29) is

$$\begin{aligned} &\left\langle 0 \left| e^{i\int_0^L \mathrm{d}y(p,\phi(y))} \right| 0 \right\rangle = \\ &= \langle 0|0\rangle + \sum_k \frac{(i)^2}{2!}\int_0^L\int_0^L \\ &\langle 0 \left|u^i(y_1)\right| k >< k \left|u^j(y_2)\right| 0 > p_i p_j \mathrm{d}\,y_1 \mathrm{d}\,y_2 \dots \end{aligned} \tag{4.30}$$

assuming that $\langle 0\,|\phi|\,0\rangle = 0$. In the infrared limit the first eigenvalue over the ground state dominates and the series in (4.30) can be summed. According to the physical interpretation previously given for the coordinate representation of (4.29) in terms of magnetic poles, in our case we have two possibilities: if the fundamental two-cycle is a two-sphere, the spectrum λ_k is

$$\lambda_k = \alpha\sum_{r=1}^{n}\ell_r\left(\ell_r + \frac{1}{2}\right), \qquad \lambda_r = \frac{1}{2}, 1, \frac{3}{2}\dots$$

because the eigenfunctions are the tensor product of the n monopole harmonics

$$\psi_k = \bigotimes_{r=1}^{n} \mathrm{D}^{\ell_r}_{m_r,\frac{1}{2}}(\vartheta_r,\varphi_r), \qquad m_r = -\ell_r, -\ell_{r+1}, \dots, \ell_{r-1}, \ell_r\,.$$

For the two-torus the spectrum and the eigenfuctions are

$$\lambda_k = \alpha\sum_{r=1}^{n}\left(\ell_r + \frac{1}{2}\right),\ \ell_r \in \mathbb{Z};$$

$$\psi_k = \bigotimes_{r+1}^{n} e^{i\left(m_r^1\vartheta_1^r + m_r^2\vartheta_2^r\right)}, \quad m_r^1 + m_r^2 = \ell_r\,.$$

In both cases the ground state is $2n$-times degenerated, $-\ell_r = \frac{1}{2}$, $m_r = \pm\frac{1}{2}$ for S^2 and $\ell_r = -1$, $\ell_r = 0$ for T^2 and the e^μ can be realized as γ^μ-matrices in the Hilbert space of dimension $2n$. Then the correlation function in the right hand side of (4.30) is constant and the answer is

$$\langle e^{i\int_0^1 \mathrm{d}x(p,\phi(x))}\rangle = e^{ip_\mu \gamma^\mu L} \tag{4.31}$$

where $\gamma^\mu = E^\mu_a \gamma^a$, $a = 1,2,\ldots,2n+1$ are flat indices and the E^μ_a are the inverse of the $(2n+1)$-beins on M.

In the functional integral (4.24) there is only one remaining integration to make: the sum over the einbeins $h(x)$. The integral is invariant with respect to one-dimensional general covariance, $x' = f(x)$. We can use this freedom to fix the gauge $h(x) = L$. $\mathbf{Z}$ is then reduced to the integral

$$\mathbf{Z} = \int \mathrm{d}L e^{-mL} e^{ip_\mu \gamma^\mu L} \tag{4.32}$$

and the propagator is the propagator of a spin $-\frac{1}{2}$ particle in a curved space because it is the inverse of the Fourier transform of the Dirac operator

$$\mathrm{D} = i\gamma_\mu \nabla^\mu + m; \qquad \nabla^\mu = \frac{\partial}{\partial X^\mu} + \frac{1}{4}[\gamma_a, \gamma_b]\omega^{\mu a b}. \tag{4.33}$$

Higher values of ℓ_r would lead to propagators for higher spins. Note that taking large L and large k simultaneously is compatible with small k_0 and, hence, with small spin.

Another important observation is the following: the large L limit is the large α limit. When $\alpha \to \infty$ only the ground states do not decouple and are important, just as in the infrared limit. This gives a clue about why the partition function of the WZW topological quantum mechanics is related with the index theorem. The recent proofs of the index theorem are based in the quantum mechanics of the sigma model [8] because the contribution of the non-zero energy states cancel due to supersymmetry. WZW topological quantum mechanics are the limit of the previous model where the non-zero eigenvalues go to infinity and give no contribution without the need of supersymmetric (fantastic) cancellations. We are left only with the ground states: no wonder the proof of the index theorem in the framework of topological quantum mechanics! An intriguing question remains: is there any deep reason for the origin of the Dirac operator when a spin factor required by the index theorem is included in the quantum action of a relativistic particle?

5. AN EXACT EXAMPLE

As a final computation we can exactly solve the case where M is the two-torus T^2, a simple but far from trivial example. The important object is the «kernel» given by the

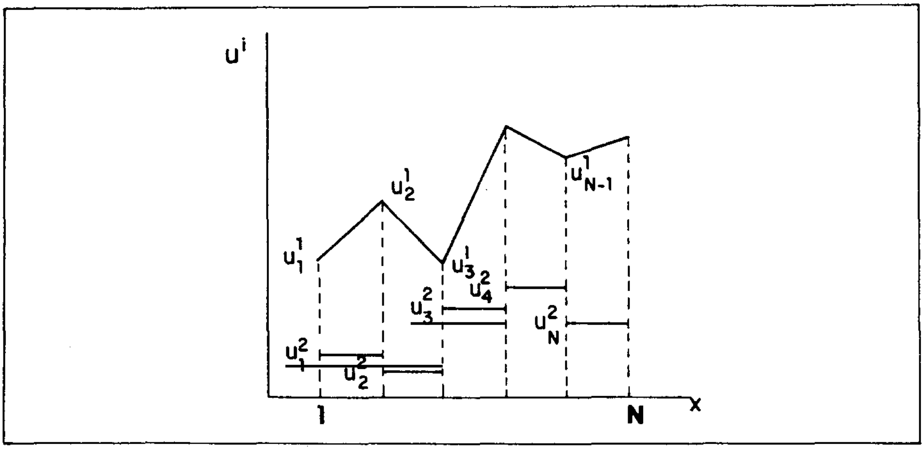

Fig. 12. Piece-wise linear and piece-wise constant trajectories in phase space.

functional integral

$$K(u_a^i, u_b^i; k) = \int [\mathrm{d}u^i] e^{ik\int_0^1 \mathrm{d}x \varepsilon_{ij}\left(u^i \frac{\mathrm{d}u^j}{\mathrm{d}x} - u^j \frac{\mathrm{d}u^i}{\mathrm{d}x}\right)} \tag{5.1}$$

$$u^i(0) = u_a^i, \qquad u^i(1) = u_b^i \qquad i = 1, 2\ ,$$

where the «coordinates» u^i parametrize a two-torus, i.e. $u^i \in [0, 1]$ and are periodic. The partition function is then

$$\mathbf{Z}(k) = \int_0^1 \mathrm{d}u_0^1 \int_0^1 \mathrm{d}u_0^2 K(u^i, u^i; k)\ . \tag{5.2}$$

To define the functional integral as a limit of finite-dimensional integrals we choose a «cartesian» polarization, u^i playing the rôle of coordinate and u^2 being the momentum. this allows us to take a polygonal approximation for the «trajectories» in phase space with u^2 being piece-wise constant and u^1 piece-wise linear, Fig. 12. The functional integral is then defined to be the limit of the product of finite-dimen- sional integrals on the values of u^i at the dividing points when the interval tends to zero. Due to the periodicity of the variable u^1 we must sum over trajectories incoming into the points $u_a^1 - m$ where m is an integer, i.e. (see Ref. [35])

$$\begin{aligned} K(u_a^i, u_b^i; k) &= \lim_{N\to\infty} \frac{1}{(2\pi)^N} \int \mathrm{d}u_1^1 \mathrm{d}u_1^2 \ldots \mathrm{d}u_{N-1}^1 \mathrm{d}u_{N-1}^2 \\ &\exp\left\{iku_1^2(u_1^1 - u_a^1) + iku_2^2(u_2^1 - u_1^1) + \ldots\right. \\ &\left.\ldots + iku_b^2(u_b^1 + m - u_{N-1}^1)\right\}\ . \end{aligned} \tag{5.3}$$

The integrals in the u^2 variables give δ-functions which allows us to integrate in u^1 to get

$$(5.4)\qquad \begin{aligned} K(u_a^1, u_b^1; k) &= \int_0^1 \mathrm{d}\, u_b^2 K(u_a^i, u_b^i; k) \\ &= \frac{1}{2\pi} \sum_m \int_0^1 \mathrm{d}\, u^2 \exp\left\{iku^2(u_b^1 - u_a^1 + m)\right\} \end{aligned}$$

fixing $u_a^2 = 0$ and integrating on $u^2 - b$. The identity

$$(5.5)\qquad \sum_m \exp\, ikmu^2 = \sum^n \delta\left(\frac{ku^2}{2\pi} - n\right) = \frac{2\pi}{k} \sum_n \delta\left(u^2 - \frac{2\pi}{k} n\right)$$

leads to

$$(5.6)\qquad K(u_a^1, u_b^1; k) = \frac{1}{k} \int_0^1 \mathrm{d}\, u^2 \exp iku^2(u_a^1 - u_b^1) \sum_n \delta\left(u^2 - \frac{2\pi}{k} n\right) .$$

Formula (5.6) yields periodic physical predictions if the «modulus» is invariant under the transformation $u'^2 = u^2 + q$ for q an integer. This gives the quantization condition $k = 2\pi q$, slightly different to that previously considered throughout the paper because we were dealing with the case of two-sphere as a fundamental two-cycle. The final answer is

$$(5.7)\qquad K(u_a^1, u_b^1; k) = \frac{1}{k} \sum_{s=0}^{q-1} \exp\, i2\pi s(u_a^1 - u_b^1) .$$

In «energy» representation

$$K(u_a^1, u_b^1; k) = \frac{1}{k} \sum_{s=0}^{q-1} \psi_{or}^*(u_b^1) \psi_{or}^*(u_a^1)$$

it is easy to see that the Hilbert space is subtended by the wave functions (ground states)

$$(5.8)\qquad \psi_{os}(u^1) = \frac{1}{\sqrt{k}} e^{i2\pi s u^1}, \qquad s = 0, 1, \ldots, q-1$$

then being finite-dimensional. In fact WZW topological quantum mechanics are in this case nothing but a $\mathbb{Z}_q, k = 2\pi q$, model on a point! Alternatively, it can be seen as the usual $\mathbb{Z}_q$ model of statistical mechanics where the block spin transformation is iterated until reducint the lattice to a point.

It is immediate to generalize this result to a Riemann surface of genus g and bundles with structure group a abelian group T, instead the $u(1)$-bundles over T^2 previously considered. The dimension of the Hilbert space in this case is $\left(\frac{k}{2\pi}\right)^{g\dim T}$. This is interesting because the prediction of the Geometric Quantizatio procedure in the SPA is exactly the same: in the topological quantum mechanics for the torus the stationary phase approximation is exact. In fact, the partition function can be exactly found from (5.4) - (5.6). We have been, however, too restrictive. The states (5.8) are periodic in u^1 but in quantum mechanics rays, rather than vectors, in Hilbert spaces are meaningful. A phase $\alpha(x)$ is allowed yielding the most general periodicity condition $\psi(u^1+1) = \alpha\psi(u^1)$, with $\alpha = \alpha(1) - \alpha(0)$. Choosing the same phase in all the terms in the energy eigenfunction expansion of K, to satisfy the general periodicity condition we get

(5.9) $$\mathbf{Z} = \sum_{s=0}^{q-1} \frac{\alpha}{k} = \frac{\alpha}{2\pi},$$

and the partition function is independent of k! It is convenient to express the partition function in the form

(5.10) $$\mathbf{Z} = \langle\psi_0|\psi_0\rangle = \frac{1}{k}\sum_{s,r}\alpha C_r^* C_s \int \mathrm{d}\,u^1 e^{i2\pi(s-r)u^1} = \frac{\alpha}{2\pi},$$

where $\left|\psi_0\right> = \sum_{s=0}^{q-1} c_s \left|\psi_{os}\right>$ and $|C_s| = 1$, because this is the way in which the topological invariant arising in QFT are axiomatized.

We can also introduce states in «momentum» representation, related with the previous one by Fourier transformations. These «spin waves» are

$$\tilde{\psi}_{os} = \frac{1}{\sqrt{k}}\frac{1}{2\pi}\int \mathrm{d}\,u^1 e^{i(2\pi s - u^2)u^1} = \frac{1}{\sqrt{k}}\delta\left(u^2 - \frac{2\pi}{k}s\right).$$

To obey the periodicity condition in u^2 $\tilde{\psi}_{os}\left(u^2 + \frac{2\pi}{k}\right) = \alpha\tilde{\psi}_{os}(u^2)$ we write states in the form

(5.11) $$\tilde{\psi}_{os}(u^2) = \frac{1}{\sqrt{k}}\sum_n \alpha^n \delta\left(u^2_+ \frac{2\pi}{k}(n+s)\right)$$

and the invariant $\langle\psi_0|\psi_0\rangle$ is also obtained by using them

$$\mathbf{Z} = \sum_{s,r}\int \mathrm{d}\,u^2 \tilde{\psi}^*_{os}(u^2)\tilde{\psi}_0(u^2)\tilde{C}^*_s\tilde{C}_r = \frac{\alpha}{2\pi}.$$

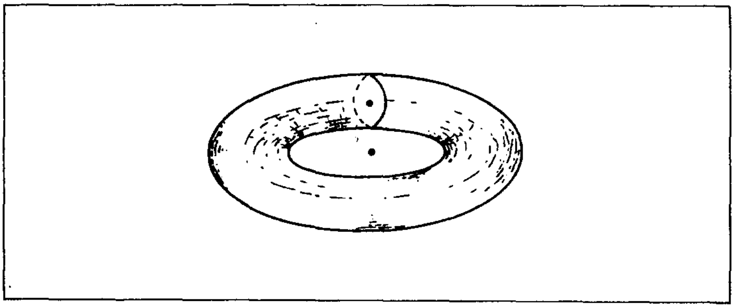

Fig. 13. Two «plane» poles creating a constant magnetic field on the torus.

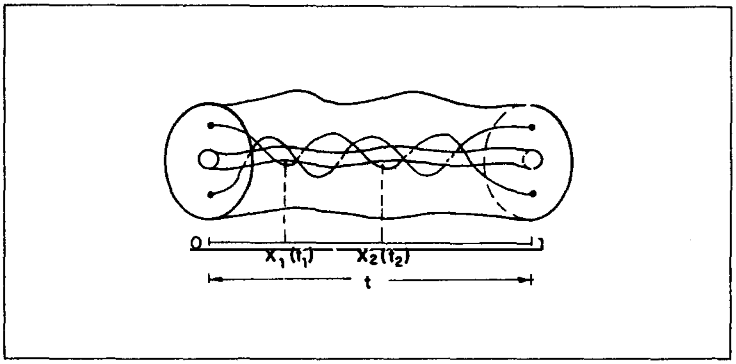

Fig. 14. Graph of β_2 in $T^2 \times I$.

A physical comment: this partition function corresponds to the spin factor of a charged particle moving on a torus under a megnetic field created by «plane» magnetic poles located at the centres of the generating one-cycles, see Fig. 15.

Finally, we shall consider braids and links, Fig. 14-15, on T^2 . In the case of two point-sources, the phase space is $T^2 \otimes T^2 - \mathrm{D}/S_n$, the braid invariant $Tr\hat{\beta}_2(T_f)/Tr\,\hat{\beta}_2(T_i)$, where

$$\text{(5.12)} \qquad \begin{aligned} Tr\hat{\beta}_2(t) &= \langle\psi_0 \left|\hat{\beta}_2(t)\right| \psi_0\rangle = \langle\psi_0|P\left(e^{ik\int dx \sum_{\alpha=1}^{n} p_{\alpha a}(x) u_\alpha^a}\right)|\psi_0\rangle \\ p_{\alpha a}(x) &= b_{\alpha a}\delta(x - x_\alpha(t)) \ , \end{aligned}$$

can be computed by the Baker-Haussdorf formula (note the path ordering necessary in the series expansion of the exponental because the operators $\hat{u}_{\alpha a}$ do not commute),

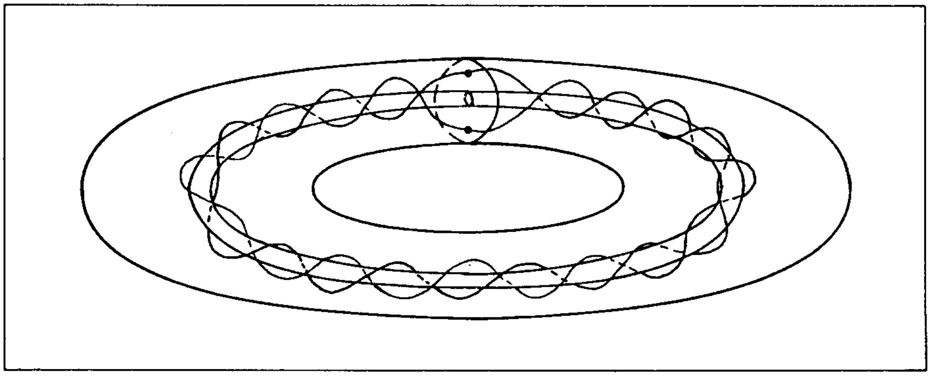

Fig. 15. The link L closing β_2 .

yielding

$$P\left(e^{ik\int \mathrm{d}x \sum_\alpha p_{\alpha a}\hat{u}^a_\alpha}\right) = \tag{5.13}$$
$$= ^{-\frac{k^2}{2}\int \mathrm{d}x \int \mathrm{d}x' \sum_\alpha \sum_\beta (p_{\alpha a}p_{\beta b} - p_{\alpha b}p_{\beta a})\left[\hat{u}^a_\alpha(x), \hat{u}^b_\beta(x')\right]}$$
$$\cdot\, e^{ik\int \mathrm{d}x \sum_\alpha p_{\alpha a}\hat{u}^a_\alpha}$$

such that

$$\langle\psi_0\left|\hat{\beta}_2\right|\psi_0\rangle = e^{i\frac{k}{2}b_1 b_2 \varepsilon(x_1(t)-x_2(t))} . <\psi_0\left|e^{ik\int \mathrm{d}x \sum_\alpha p_{\alpha a}\hat{u}^a_\alpha}\right|\psi_0> \tag{5.14}$$
$$b_\alpha = b_{\alpha 1} + ib_{\alpha 2} \, .$$

To compute $\langle\psi_0\left|e^{ik\int \mathrm{d}x \sum_\alpha p_{\alpha a}\hat{u}^a_\alpha}\right|\psi_0\rangle$, one can use the unit projector in terms of the eigenstates of the $\hat{u}^a s$ and the final answer is $\alpha/2\pi$, if a phase α is allowed in the state $|\psi_0>$.

For the link-invariant a functional integral is better suited. The partition function is

$$\mathbf{Z}(k; L) = \int \mathrm{d}u^0_{\alpha a} \int_{u_{\alpha a}[x(T_i)]=u^0_{\alpha a}}^{u_{\alpha a}[x(T_f)]=u^0_{\alpha a}} \tag{5.15}$$
$$[\mathrm{d}u_{\alpha a}]e^{ik\int \mathrm{d}x\left[\sum_\alpha \varepsilon_{ab}\left(u^a_\alpha \frac{\mathrm{d}u^b_\alpha}{\mathrm{d}x} - u^b_\alpha \frac{\mathrm{d}u^a_\alpha}{\mathrm{d}x}\right)\right]} e^{ik\int \mathrm{d}x \sum_\alpha p_{\alpha a}u^a_\alpha} \, .$$

By changing the integration variables to $u^a_\alpha = v^a_\alpha - \int \mathrm{d}x \mathrm{D}^{ab}(x-x')p_{\alpha b}(x')$, where $\mathrm{D}^{ab}(x-x') = \varepsilon^{ab}\varepsilon(x-x')$ is the Green function for the differential operator in the exponential of (5.15)

$$\varepsilon_{ab}\frac{\mathrm{d}\mathrm{D}^{ab}}{\mathrm{d}x}(x-x') = \delta(x-x') \tag{5.16}$$

one easily obtains:

$$\mathbf{Z}(k;L) = \mathbf{Z}(k)e^{j\frac{k}{2}b_1^*b_2[\varepsilon(x_1(T_f)-x_2(T_f)-\varepsilon(x_1(T_i)-x_2(T))]} \tag{5.17}$$

and the result (5.14) is recovered.

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